A local-global principle for the real continuum

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Abstract We discuss the implications of a *local-global* (or *global-limit*) principle for proving the basic theorems of real analysis. The aim is to improve the set of available tools in real analysis, where the local-global principle is used as a unifying principle from which the other completeness axioms and several classical theorems are proved in a fairly direct way. As a consequence, the study of the local-global concept can help establish better pedagogical approaches for teaching classical analysis.

1 Introduction

The logical foundations of mathematical analysis were developed at the end of 19th century and beginning of 20th century by mathematicians such as B. Bolzano ($\sim$1817), A. L. Cauchy ($\sim$1821–1829), K. Weierstrass ($\sim$1865-1895), C. Méray ($\sim$1869), R. Dedekind ($\sim$1872), G. Cantor ($\sim$1872), E. Heine ($\sim$1872), E. Borel ($\sim$1895-1903), P. Cousin ($\sim$1895) and H. Lebesgue ($\sim$1905). They departed from the geometric intuition of the "real line" by establishing rigorous proofs based on *completeness axioms* that characterize the real number continuum.

As noticed in [4, 14, 27], rigor was not the most pressing question. Instead these authors focused on *teaching*. Several mathematicians found themselves in an awkward situation when they had to teach differential and integral calculus based on fuzzy geometric evidences. Therefore, they decided to reform it [27]. Examples are Cauchy's Cours d'Analyse at École Polytechnique in Paris, Weierstrass's lectures at the University of Berlin and Dedekind's course at Zürich Polytechnic. Dedekind wrote:

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> *In discussing the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continually, but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences.(...) For myself this feeling of dissatisfaction (...) I should find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis. [8, pp.1-2]*

Felix Klein coined the phrase "the arithmetizing of mathematics" [22], a classic of the era of rigor. Until today, the foundation has not been put into question; it is still recognized as satisfactory in all classical textbooks which define $\mathbb{R}$ as any ordered field satisfying one of the equivalent *completeness axioms* listed below[1]:

Sup (Least Upper Bound Property) Any set of real numbers has a supremum (and an infimum)[2];
Cut (Dedekind's Completeness) Any cut defines a (unique) real number;
Nest+Arch (Cantor's Property) Any sequence of nested closed intervals has a common point + Archimedean property;
Cauchy+Arch (Cauchy's Completeness) Any Cauchy sequence converges + Archimedean property;
Mono (Monotone Convergence) Any monotonic sequence has a limit[2];
BW (Bolzano-Weierstrass) Any infinite set of real numbers (or any sequence) has a limit point[2];
BL (Borel-Lebesgue) Any cover of a closed interval by open intervals has a finite subcover[3];
Cousin (Cousin's partition [12]) Any gauge defined on a closed interval admits a fine tagged partition of this interval;
Ind (Continuous Induction [5, 17, 20]).

One may find it striking that all these equivalent properties look so diverse. This calls for the need of a simple unifying principle from which all such properties could be easily and directly derived as theorems. In this article, we introduce and discuss two versions of yet another equivalent axiom:

LG (Local-Global) Any *local* and *additive* property is *global*;
GL (Global-Limit) Any *global* and *subtractive* property has a *limit point*.

The earliest reference we could find that explicitly describes this principle is Guyou's little-known French textbook [16]. Guyou wrote:

> *Les démonstrations de ce livre sont, en général, différentes des démonstrations classiques; un tel remaniement comporte sans doute des erreurs, que je serai reconnaissant à mes collègues de bien vouloir me signaler. [16, p. xv].*
>
> *[The proofs in this book are, in general, different from the classical proofs; such a reworking may contain errors, that I shall be grateful to my colleagues for pointing out to me.]*

[1] Precise definitions will be given in Section 4. Some of the statements require the Archimedean property: Any real number is upper bounded by a natural number.

[2] Possibly infinite, e.g., $\sup \mathbb{R} = \inf \varnothing = +\infty$, $\sup \varnothing = \inf \mathbb{R} = -\infty$, $\lim \pm n = \pm\infty$.

[3] This is Borel's statement, also (somewhat wrongly) attributed to Heine, and later generalized by Lebesgue and others [1].

The principle was later re-discovered independently and frequently in many disguises in some American circles [11, 21, 26, 28, 30]. One central concept is the notion of interval-additive property set up independently by Guyou and Ford [11, 16], which we feel can be useful for pedagogical purposes:

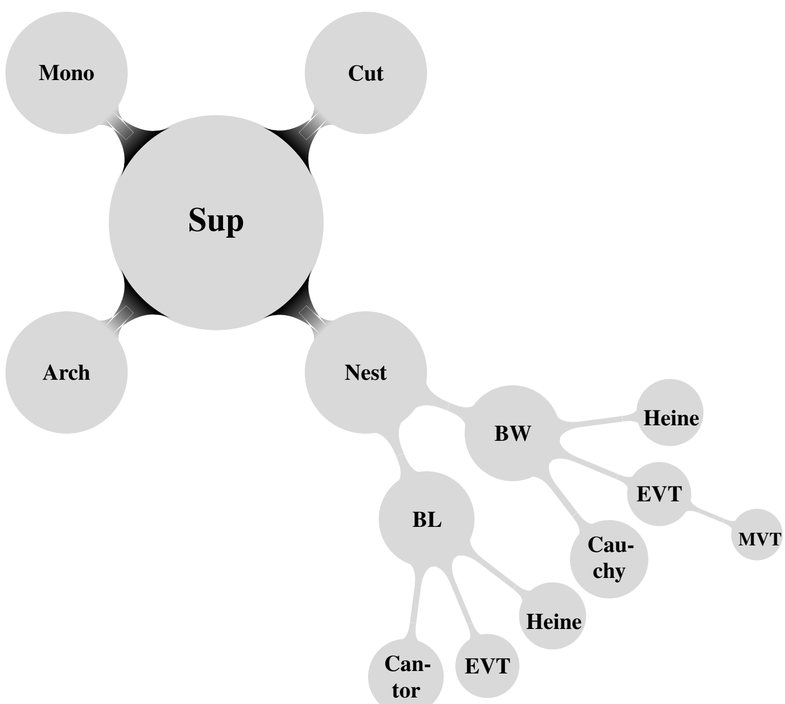
> *A statement P concerning intervals will be called interval-additive if whenever P is true for each of two overlapping intervals [. . .] it is also true for the interval obtained by combining them; that is, their union. [11, p. 106]*

2 Teaching Real Analysis: Present situation

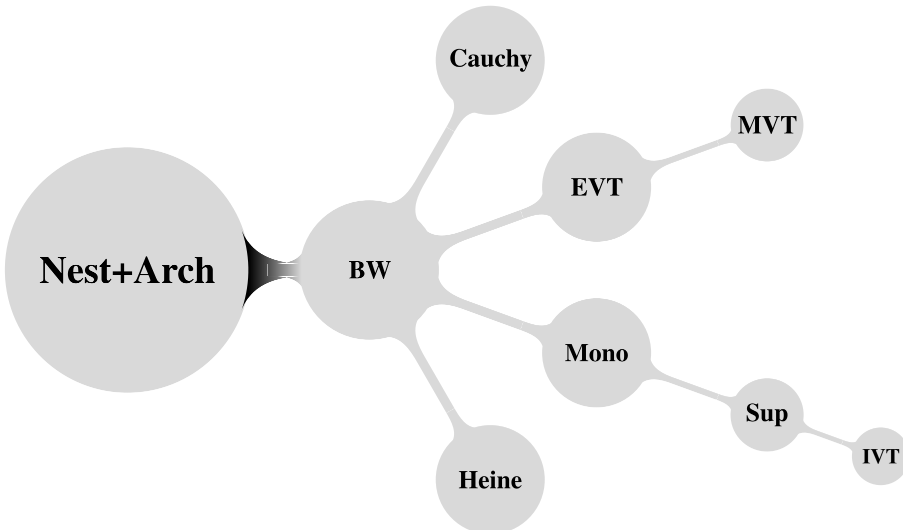
We have studied the logical flow of proofs in detail in the most influential undergraduate/graduate textbooks in the U.S.A. [2, 7, 29], France [9, 25] and Brazil [15, 24]. These included not only proofs of the essential properties of the real numbers, but also of the basic theorems for continuity (boundness theorem **BT**, intermediate value theorem **IVT**, extreme value theorem **EVT**, Heine's uniform continuity theorem **Heine**) and differentiation (essentially the mean value theorem **MVT**).

We identified the logical flows of each of these textbooks as follows:

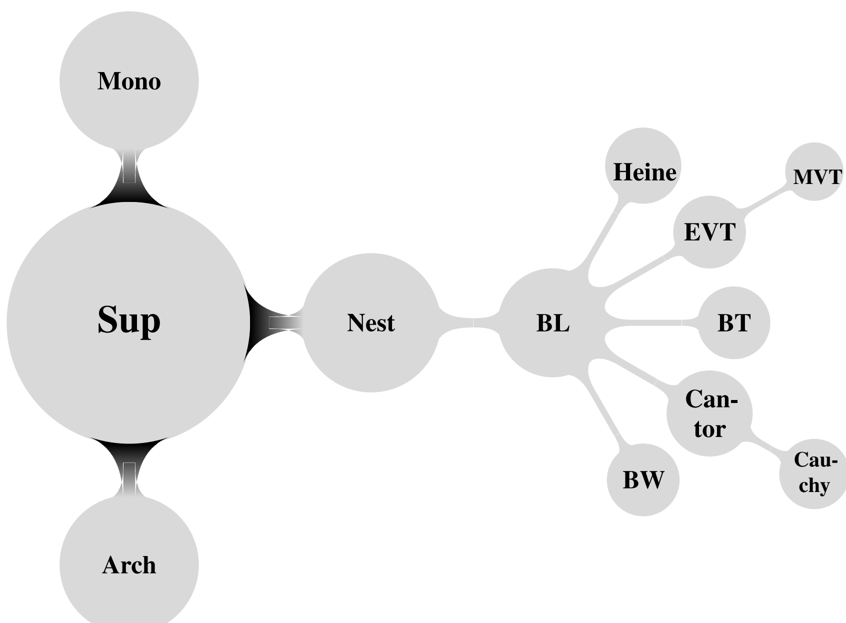
Robert G. Bartle, Elements of Real Analysis [2]



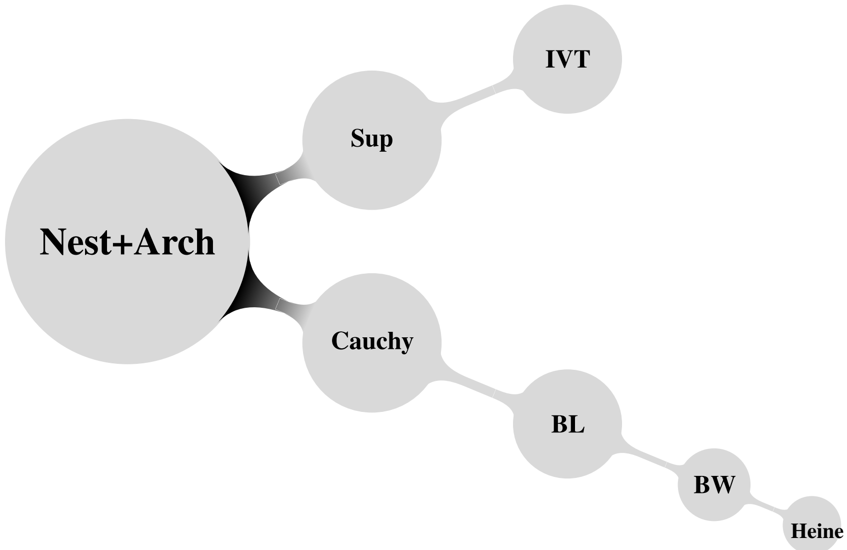
Richard Courant, Introduction to Calculus and Analysis I [7]



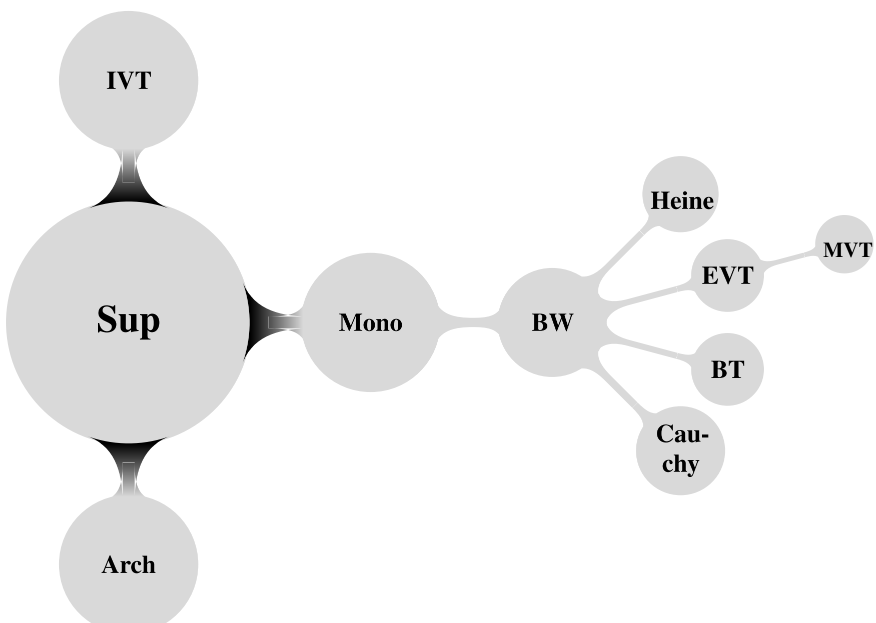
Walter Rudin, Principles of Mathematical Analysis [29]



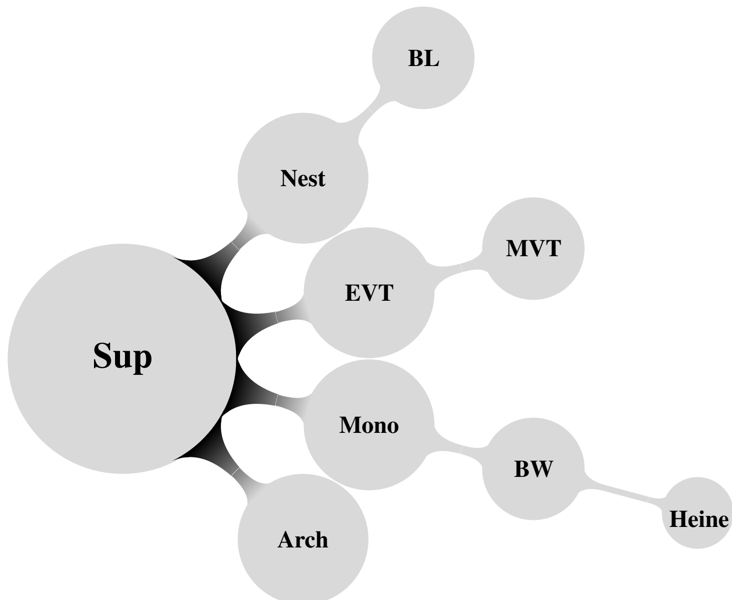
Jean Dieudonné, Foundations of Modern Analysis [9]



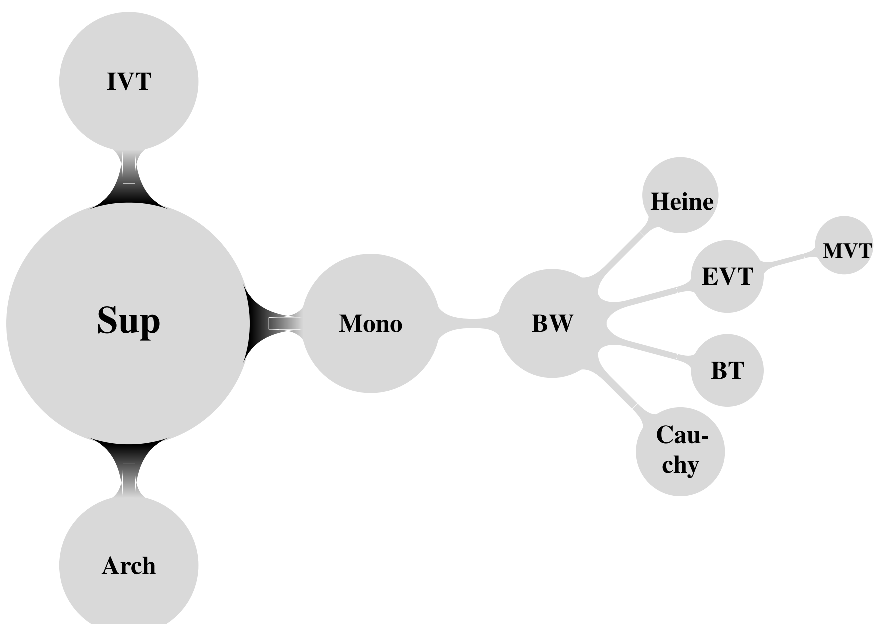
Serge Lang, Undergraduate Analysis [25]



Elon Lages Lima - Análise Real [24]



Djairo Guedes de Figueiredo - Análise I [15]

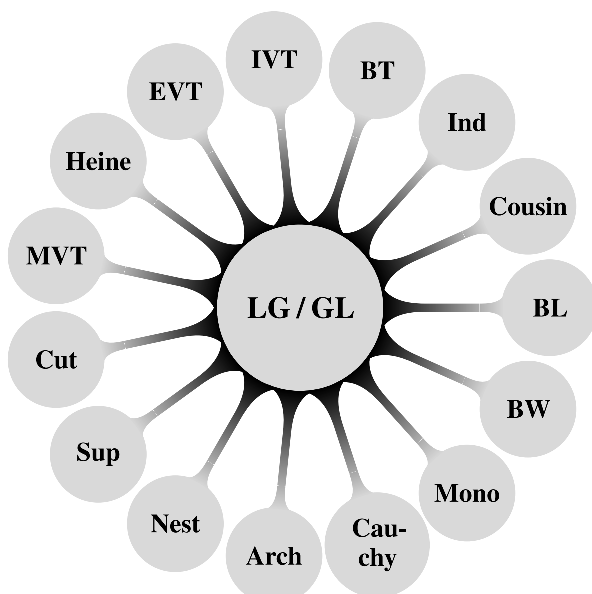


It appears that **Sup** is by far the preferred axiom, with **Nest+Arch** being the only considered alternative in [7, 9]. Other axioms (**Cauchy, Arch**, **Mono**, **BW**, often **BL**, and sometimes **Cut**) are derived as theorems. In contrast, **Cousin** and **Ind** are never used[4]. **BW** is often central to prove the basic theorems of real analysis (particularly **IVT**, **EVT**, **Heine**) with **BL** sometimes used as a "topological" alternative.

Our conclusion about classical approaches is that several classical proofs are quite difficult and subtle for the beginner (e.g., proofs of **EVT** or **Heine** using **BW**). Recent attempts to improve this situation in the literature advocate the use of **Cousin** [12] or **Ind** [5, 17], but this can also be cumbersome at times (although we agree that this is a matter of opinion).

Some textbooks (such as [29]) also mention the possibility of *proving* the fundamental axiom by first *constructing* the reals from the rationals—themselves constructed from the natural numbers— the two most popular construction methods being Dedekind's cuts and Cantor's fundamental sequences. While this is satisfactory for logical consistency, the details are always tedious and not very instructive for the student or for anyone using the real numbers, since the way they can be constructed never influences the way they are actually used.

In the following sections we describe the **LG/GL** alternative, which we show is one basic unifying principle from which all other completeness axioms and basic continuity and differentiability theorems are easily derived, as illustrated in the following figure. In this way a teacher may advantageously choose to teach (or not to teach) any given item.



[4] Although proposed at the same time as Borel's **BL**, Cousin has been largely overlooked since. It was only recently re-exhumed as a fundamental lemma for deriving the gauge (Kurzweil-Henstock) integral (e.g. [12]). **Ind** is much more recent and in fact inspired from **LG** (see [10, 19]).

3 The Local-Global Principle: A Primer

In the remainder of this paper (perhaps with the exception of section 5), our presentation is deliberately at the simplest undergraduate level. In particular, we do not use explicitly topological concepts such as compactness and connectedness (even though these could be easily addressed within the present framework) and start with what we think of as the simplest type of point sets, namely intervals. We also stay one-dimensional although the concepts derived here can be easily generalized to point sets in any dimension.

Let us explain the above **LG** and **GL** principles by defining the following intuitive notions.

3.1 Points and Intervals

We follow the classical notations of points and intervals with some non-traditional definitions which will now be explained. We add two new symbols $-\infty$ and $+\infty$ to the usual real set $\mathbb{R}$ such that:

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$$

is totally ordered where $-\infty$ and $+\infty$ are the smallest and largest elements of $\mathbb{R}$, respectively.

Definition 1. A real *point* is an element of $\overline{\mathbb{R}}$. A finite point is an element of $\mathbb{R}$, and an infinite point is $-\infty$ or $+\infty$.

We feel that introducing explicitly *infinite* points is quite convenient here because it allows for simpler statements where e.g., we can dispense with assuming that a given point set is *bounded* to prove the existence of some limit point (since that limit point can be infinite). In other words, our take on the classical debate of potential vs. actual infinite leans toward adopting actual infinities, essentially for convenience.

Most of the sets considered in the sequel will be point sets in $\overline{\mathbb{R}}$, particularly intervals. We consider two kinds of intervals:

Definition 2. Let u, v points in $\overline{\mathbb{R}}$ such that $u < v$. The following are the two kinds of intervals with u and v as extremities:

$$[u, v] = \{x \in \overline{\mathbb{R}} | u \leq x \leq v\}$$
$$]u, v[= \{x \in \overline{\mathbb{R}} | u < x < v\}$$

To simplify the assertions we consider any $[a, b] \subseteq [-\infty, +\infty] = \overline{\mathbb{R}}$ and assume that all closed intervals $[u, v] \subseteq [a, b]$ are nondegenerate ($u < v$). Again this implicit convention of nondegenerate intervals will appear quite convenient in what follows.

Definition 3. Intervals are *overlapping* if their intersection is an interval[5].

Central in our study are properties of intervals $[u,v] \subseteq [a,b]$. Guyou wrote:

> *Une fonction $f(x)$, qui possède la propriété d'être bornée (nous appellerons cette propriété P) dans deux intervalles contigus, possède la même propriété dans l'intervalle somme des deux (nous dirons que P est additive). [16, p.32]*
>
> *[A function $f(x)$, which has the property of being bounded (we shall call this property P) in two contiguous intervals, has the same property in the sum of the two intervals (we shall say that P is additive).]*

Here we shall always consider properties $\mathscr{P}$ of such intervals, and any property $\mathscr{P}$ is identified with the set of intervals $[u,v]$ that satisfies this property. Thus we write "$[u,v] \in \mathscr{P}$" if $[u,v]$ satisfies the property $\mathscr{P}$. The negation $\neg\mathscr{P}$ of property is identified to the complementary set of $\mathscr{P}$, that is, $[u,v] \notin \mathscr{P} \Longleftrightarrow [u,v] \in \neg\mathscr{P}$.

Since we identify $\mathscr{P}$ with a family of closed subintervals of $[a,b]$, the set of all properties $\mathscr{P}$ can be thought as the set-theoretic abstraction of the set of all such families of closed subintervals. It is of course much simpler to think of the statement $[u,v] \in \mathscr{P}$ as a property satisfied by an interval $[u,v]$ which is itself characterized by two endpoints $u < v$. Thus $[u,v] \in \mathscr{P}$ can be simply thought as a binary relation $u\mathscr{R}v$ on the set of ordered endpoints $u < v$.

3.2 Additive and Subtractive Properties

Definition 4. A property $\mathscr{P}$ is *additive* if for any $u < v < w$

$$[u,v] \in \mathscr{P} \ \wedge \ [v,w] \in \mathscr{P} \implies [u,w] \in \mathscr{P}.$$

A consequence of $\mathscr{P}$ being additive is that it defines a transitive binary relation $u\mathscr{R}v \Longleftrightarrow [u,v] \in \mathscr{P}$.

Definition 5. A property $\mathscr{P}$ is *subtractive* if for any $u < v < w$,

$$[u,w] \in \mathscr{P} \implies [u,v] \in \mathscr{P} \ \vee \ [v,w] \in \mathscr{P}.$$

Proposition 1. *A property $\mathscr{P}$ is* additive *if and only if its negation $\neg\mathscr{P}$ is subtractive.*

Proof. The contraposition of statement $[u,v] \in \mathscr{P} \wedge [v,w] \in \mathscr{P} \implies [u,w] \in \mathscr{P}$ is $[u,w] \in \neg\mathscr{P} \implies [u,v] \in \neg\mathscr{P} \vee [v,w] \in \neg\mathscr{P}$. □

Example 1. The property of a function being positive (or nondecreasing, or continuous) on $[u,v]$ is additive and subtractive (and in fact true for any subinterval).

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[5] A non-degenerate interval. Hence two adjacent intervals $[u,v]$ and $[v,w]$ (where $u < v < w$) are not overlapping since their intersection is reduced to a point.

Example 2. The property that $[u,v]$ has exactly one integer is subtractive but not additive. The property that $[u,v]$ has at least two integers is additive but not subtractive.

Example 3. An important example of a quite general subtractive property is "$u \notin E$ and $v \in E$" where E is any set of points. In fact, if $u < v < w$, and if $[u,w] \in \mathscr{P}$, we have $[u,v] \in \mathscr{P}$ or $[v,w] \in \mathscr{P}$ according to $v \in E$ or not. As an example, for all real function f, the property "$f(u) \leq 0$ and $f(v) > 0$" is subtractive. Similarly, the property "$f(u) \leq 0$ and $f(v) \geq 0$" is also subtractive.

Certain properties are true under slightly different conditions that require a partial overlap between the two subintervals:

Definition 6. A property is *o-additive* (overlap-additive) if for all $t < u < v < w$,

$$[t,v] \in \mathscr{P} \wedge [u,w] \in \mathscr{P} \implies [t,w] \in \mathscr{P}.$$

Example 4. The property "$(v-u) \geq 1$" is additive and *o*-additive. The property "$(v-u) \leq 1$" is neither additive, nor *o*-additive.

Example 5. The property "f is a linear function on $[u,v]$" is *o*-additive, but not additive (consider a piecewise linear function). The property "f is a convex function on $[u,v]$" is *o*-additive, but not additive.

Example 6. The property "$f(u)f(v) > 0$" ("$f(u)$ and $f(v)$ are of the same sign") is additive, but not *o*-additive.

Definition 7. A property $\mathscr{P}$ is *o-subtractive* if for all $t < u < v < w$,

$$[t,w] \in \mathscr{P} \implies [t,v] \in \mathscr{P} \vee [u,w] \in \mathscr{P}.$$

Example 7. Any property such that, when true for an interval, remains true for any subinterval ($[u',v'] \subset [u,v] \in \mathscr{P}$ implies $[u',v'] \in \mathscr{P}$) is *a fortiori* *o*-subtractive.

Example 8. The property "$(v-u) < 1$" is subtractive and *o*-subtractive. The property "$(v-u) > 1$" is neither subtractive nor *o*-subtractive.

Proposition 2. *A property $\mathscr{P}$ is o-additive if and only if its negation $\neg\mathscr{P}$ is o-subtractive.*

Proof. The contraposition of "$[t,v] \in \mathscr{P} \wedge [u,w] \in \mathscr{P} \implies [t,w] \in \mathscr{P}$" is "$[t,w] \in \neg\mathscr{P} \implies [t,v] \in \neg\mathscr{P} \vee [u,w] \in \neg\mathscr{P}$". □

Example 9. The property "f is a nonlinear function on $[u,v]$" is *o*-subtractive but not subtractive.

Example 10. The property $f(u)f(v) \leq 0$ ("$f(u)$ and $f(v)$ have opposite signs") is subtractive but not *o*-subtractive.

3.3 Neighborhoods

Instead of using the general notion of a neighborhood in an abstract topological space we use the following equivalent notion for $\overline{\mathbb{R}}$, based on intervals, which is enough for our purposes. A neighborhood $V(x)$ of a point $x \in [a,b]$ contains all points "sufficiently close" to x:

Definition 8. A neighborhood $V(x)$ of a point $x \in [a,b]$ is any set of points containing at least one interval $[u,v]$ such that:

$$\begin{cases} u < x < v, & \text{if} \quad a < x < b; \\ a = u < x, & \text{if} \quad x = a; \\ u < v = b, & \text{if} \quad x = b. \end{cases}$$

Thus in a neighborhood $V(x)$, it is possible to approach x from both sides if x belongs to the interior of $[a,b]$, but only from one side if x is one of the extremities of $[a,b]$. The notion of neighborhood depends on the considered set $[a,b]$. In most situations one may consider only neighborhoods that are themselves intervals.

Definition 9. An interval $[u,v]$ is *adapted* to neighborhood $V(x)$ if $x \in [u,v] \subseteq V(x)$.

3.4 Local Properties and Limit Points

Definition 10. A property $\mathscr{P}$ is *local* at x if there exists a neighborhood $V(x)$ such that all intervals adapted to x satisfy $\mathscr{P}$, i.e.,

$$\exists V(x), \forall [u,v] \text{ adapted to } V(x), [u,v] \in \mathscr{P}.$$

A property $\mathscr{P}$ is local on a set of points E if it is local at any point in E.

Example 11. As will be seen later, continuity and differentiability of functions are local properties. For example, a function f is continuous iff for any $\varepsilon > 0$, "$|f(u) - f(v)| < \varepsilon$" is local. Some topological properties such as interior point or isolated point can also be seen as local properties.

Definition 11. A property $\mathscr{P}$ has a *limit* point x if each neighborhood $V(x)$ contains an adapted interval which satisfies $\mathscr{P}$, i.e.,

$$\forall V(x), \exists [u,v] \text{ adapted to } V(x), [u,v] \in \mathscr{P}.$$

A property $\mathscr{P}$ has a *limit* on a set of points E if it has a limit at each point of E. It can be easily seen as an exercise that any property local at x does have a limit at x.

Proposition 3. *A property $\mathscr{P}$ is not* local *at x if and only if its negation $\neg\mathscr{P}$ has a limit at x.*

Proof. The negation of the assertion $\exists V(x), \forall [u,v]$ adapted to $V(x), [u,v] \in \mathscr{P}$ is $\forall V(x), \exists [u,v]$ adapted to $V(x), [u,v] \in \neg\mathscr{P}$. ☐

Thus to say that $\mathscr{P}$ is *not* local on a set E is the same as saying that $\neg\mathscr{P}$ has a limit at *at least* one point of E.

3.5 Local-Global and Global-Limit Axioms

We introduce the following as foundations (completeness axioms) for the real numbers. As shown later in Sections 4 and 5 (Prop. 9), any one of these axioms is enough to characterize $\mathbb{R}$ or $\overline{\mathbb{R}}$ just as it is done traditionally by other completeness axioms[6]. Again let $[a,b]$ be any interval in $\overline{\mathbb{R}}$.

Local-Global Axiom (LG). *Every* local *and* additive *property on* $[a,b]$ *is* global*, that is, satisfied by* $[a,b]$.

Global-Limit Axiom (GL). *Every* global *and* subtractive *property has a* limit *point in* $[a,b]$.

Proposition 4. *The LG and GL axioms are equivalent.*

Proof. Let $\mathscr{P}$ be additive, that is, $\neg\mathscr{P}$ is subtractive. The LG axiom can then be written as: if $\mathscr{P}$ is local in $[a,b]$ then $[a,b] \in \mathscr{P}$. This is in turn equivalent by contraposition to the statement: if $[a,b] \in \neg\mathscr{P}$ then $\neg\mathscr{P}$ has a limit point in $[a,b]$, which is the GL axiom. ☐

Lemma 1. *Any property that is both local and o-additive is additive.*

Proof. Suppose $[u,v] \in \mathscr{P}$ and $[v,w] \in \mathscr{P}$. Since $\mathscr{P}$ is local at v, there exists a neighborhood of v in which any interval $[r,s]$ which contains v satisfies $\mathscr{P}$. We may then assume that $u < r < v < s < w$. Since $\mathscr{P}$ is o-additive, $[u,v],[r,s] \in \mathscr{P}$ implies $[u,s] \in \mathscr{P}$, then $[u,s],[v,w] \in \mathscr{P}$ implies $[u,w] \in \mathscr{P}$. Thus $\mathscr{P}$ is additive. ☐

It follows from the lemma that in the LG axiom, we may always consider either additive or o-additive properties. Thus we obtain the equivalent variants:

Local-Global Axiom (LG)–variant. *Every* local *and* o-additive *property is* global*, that is, satisfied by* $[a,b]$.

Global-Limit Axiom (GL)–variant. *Every* global *and* o-subtractive *property has a* limit *point in* $[a,b]$.

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[6] Even though the proposed axioms appear to be second-order statements since they are quantified over properties of sets, in fact any property is simply identified to a family of subintervals. Therefore, the axioms only require the basic (first order) ZF theory (with or without the axiom of choice), as is usual when teaching real analysis at an elementary level.

4 Elementary Theorems for the Reals

Many common mathematical properties can be identified as local and many common mathematical objects can be identified as limit points. For example, a function f is continuous iff for any $\varepsilon > 0$, "$|f(u) - f(v)| < \varepsilon$" is local; a sequence x_k converges iff "$x_k \in [u,v]$ for sufficiently large k" has a limit point. Thus, taking LG or GL as the fundamental completeness axiom for the real numbers it becomes easy and intuitive to prove all the other completeness properties, as well as all the basic theorems of real analysis. We start with the elementary theorems for the reals.

4.1 Dedekind Cuts

Definition 12. A Dedekind cut is a pair (E,E') where point set E and its complement set E' in $[a,b]$ are such that $E' < E$, that is, $u < v$ for any $v \in E$ and $u \notin E$.

Theorem 1 (Cut: Dedekind's Completeness). *Any cut (E,E') defines a (unique) point x such that $E' \leq x \leq E$.*

Proof. We can assume by hypothesis that E and E' are non-empty sets. The property $[u,v] \in \mathscr{P}$ with $u \notin E$ and $v \in E$ is global and also subtractive (see Example 3). By the GL axiom, $\mathscr{P}$ has a limit point x: any neighborhood $V(x)$ contains $u < x < v$ such that $u \notin E$ and $v \in E$. From this we can deduce that no point $x' \in E$ is $< x$, otherwise we could find $u \in E'$ so that $x' < u < x$, which contradicts the hypothesis $E' < E$. Similarly no point in E' is $> x$. Hence $E' \leq x \leq E$. □

For completeness we observe the following.

Proposition 5. *The LG axiom is equivalent to Dedekind's completeness.*

Proof. It is enough to prove the GL axiom from Dedekind's completeness theorem. Let $\mathscr{P}$ be global and subtractive and E be the set of points v for which $[a,v'] \in \mathscr{P}$ for all $v' \geq v$. Clearly $a \notin E$ (since $[a,a]$ is not a non-degenerate interval) and $b \in E$. Since $v \in E$ implies that all $v' \geq v$ are in E, one has $E' \leq E$ and (E,E') is a cut, and there exists x such that $E' \leq x \leq E$. In every neighborhood $V(x)$ one can find $[u,v]$ such that $[a,v] \in \mathscr{P}$ but $[a,u] \notin \mathscr{P}$. Since $\mathscr{P}$ is subtractive, $[u,v] \in \mathscr{P}$. Hence $\mathscr{P}$ has limit point x. □

4.2 Supremum and Infimum

Instead of the usual definitions of supremum (least upper bound) and infimum, we may use the following definitions which are easily shown to be equivalent and are more convenient for our purposes.

Definition 13 (Supremum). A point set E has upper bound v is $E \leq v$. It has supremum $x = \sup E$ if the property that "v is an upper bound of E and u is not an upper bound of E" has limit point x.

Lower bound and infimum are defined similarly.

Theorem 2 (Sup: Least Upper Bound Property). *Every set of points has a supremum.*

Proof. We may assume that the set E is non empty (otherwise $\sup E = -\infty$) and not reduced to $\{a\}$ (in which case $\sup E = a$). The above-mentioned property $\mathscr{P}$: "$v \geq E, u \not\geq E$" is global since $b \geq E$ and $a \not\geq E$. It is also clearly subtractive (see Example 3). By the GL axiom, $\mathscr{P}$ has a limit point which is $\sup E$. □

For completeness we observe the following.

Proposition 6. *The LG axiom is equivalent to the supremum theorem.*

Proof. It is enough to prove the LG axiom from the supremum theorem. Let $\mathscr{P}$ be local and additive and set $s = \sup\{x \mid [a,x] \in \mathscr{P}\}$. Since $\mathscr{P}$ is local in s, each interval $[u,v]$ adapted to a neighborhood $V(s)$ satisfies $\mathscr{P}$. If $s < b$, one can choose $[u,v]$ such that $v > s$ and $[a,u] \in \mathscr{P}$. Then by additivity, $[a,v] \in \mathscr{P}$ which contradicts the definition of s. Therefore $s = b$ and $[a,b] \in \mathscr{P}$. □

4.3 Continuous Induction Principle

In this section we consider properties on *points* in $[a,b]$. Such a property P can be identified with a set of points satisfying this property; then $x \in P$ is just a short notation for "x satisfies P". The principle of mathematical induction in $\mathbb{N}$ concerns the properties of natural numbers. The property $P \subset \mathbb{N}$ is *inductive* if:

1. $0 \in P$;
2. if $n > 0$ and if all $k < n$ satisfies P, then $n \in P$;
3. if $n \in P$, then $n+1 \in P$.

Due to the discrete nature of the integers, condition 2 implies condition 3, so we may only assume conditions 1 and 3 (the usual induction) or conditions 1 and 2 (the "strong induction"). The principle of mathematical induction says that any inductive property is satisfied for all integers: P inductive $\implies P = \mathbb{N}$. This principle can now be stated for the real numbers as follows:

Definition 14. A property P (subset of $[a,b]$) is *inductive* if

1. $a \in P$;
2. if $x > a$, and if all $u < x$ satisfies P, then $x \in P$;
3. if $x < b$, and if $x \in P$, then there exists $v > x$ such that $[x,v] \subset P$.

Theorem 3 (Ind: Continuous Induction (or Real Induction [5])). *Any inductive property P is global. (In other words, the only induction subset of $[a,b]$ is $[a,b]$ itself.)*

Proof. Let $\mathscr{P}$ be the property of intervals $[u,v]$ defined by "each point $\leq u$ satisfies P but there exists a point $\leq v$ which does not satisfy P". This property is subtractive (see Example 3). Since $a \in P$, if it is not true that any point $\leq b$ satisfies P, then $\mathscr{P}$ would be global. Assume by contradiction that this is the case. By the GL axiom, $\mathscr{P}$ admit a limit point x. Then all $u < x$ satisfies P and by condition 2, $x \in P$. If $x < b$, by condition 3, we find $v > x$ such that each point $\leq v$ satisfies P, which contradicts x as a limit point of $\mathscr{P}$. Therefore the limit point is equal to b, and each point $\leq b$ satisfies P. □

For completeness we observe the following.

Proposition 7. *The LG axiom is equivalent to the continuous induction principle.*

Proof. It is enough to prove the LG axiom from the continuous induction principle. Let $\mathscr{P}$ be local and additive and let P be the set of x points for which either $x = a$ or $[a,x] \in \mathscr{P}$. By hypothesis $a \in P$.

- Let $x > a$ and assume that all points $< x$ satisfy P. Since $\mathscr{P}$ is local in x, each interval $[u,v]$ adapted to a certain neighborhood $V(x)$ satisfies $\mathscr{P}$. Since $x > a$ we can choose $[u,v]$ such that $a < u < x$ and $v = x$. Then $u \in P$. Both intervals $[a,u]$, $[u,x]$ satisfy $\mathscr{P}$. Since $\mathscr{P}$ is additive, $[a,x]$ satisfies $\mathscr{P}$, so $x \in P$.
- Let $x < b$ and assume that $x \in P$. As $\mathscr{P}$ is local in x, each interval $[u,v]$ adapted to a certain neighborhood $V(x)$ satisfies $\mathscr{P}$. Since $x < b$ we can choose $[u,v]$ so that $u = x$ and $v > x$. For all v' such that $x < v' \leq v$, $[x,v']$ remains adapted to $V(x)$, and so satisfies $\mathscr{P}$. If $a = x$, we have $[a,v'] \in \mathscr{P}$. Otherwise $a < x$, and the two intervals $[a,x]$, $[x,v']$ satisfy $\mathscr{P}$. Since $\mathscr{P}$ is additive, $[a,v'] \in \mathscr{P}$. In both cases $[a,v'] \in \mathscr{P}$ for all v' such that $x < v' \leq v$, hence $[x,v] \subset P$.

By continuous induction we deduce that $b \in P$, that is, $[a,b] \in \mathscr{P}$. □

4.4 Monotone Limits

Instead of the usual definition of a limit of a sequence we can adopt the following definition which is easily shown to be equivalent and is more convenient for our purposes.

Definition 15. A sequence (x_k) of points has limit ℓ: $x_k \to \ell$ if the property $\mathscr{P}$ that "$[u,v]$ contains all x_k for large enough k" has limit point ℓ.

Notice that ℓ can be either finite or infinite.

Theorem 4 (Mono: Monotone Convergence). *Any monotonic sequence has a limit.*

Proof. Let (x_k) be a monotonic sequence of points in $[a,b]$. We may assume that (x_k) is nondecreasing (otherwise consider $(-x_k)$). The above-mentioned property "$[u,v]$ contains all x_k are large enough k" is clearly global and amounts to say that "v is greater than or equal to all x_k and u is not greater than or equal to all x_k". This property is subtractive (see Example 3). By the GL axiom, $\mathscr{P}$ has a limit point x. Therefore, $x_k \to x$. □

4.5 Archimedean Property

Theorem 5 (Arch: Archimedean Property). *$k \to +\infty$, i.e., for every $u \in \mathbb{R}$ we have $k \geq u$ for large enough k.*

As usual this implies that the set $\mathbb{Q}$ of rational numbers is dense, i.e., each interval $[u,v]$ contains a rational number.

Proof. Consider the sequence (k) of natural numbers in $[a,b] = [0,+\infty]$ and let $[u,v] \in \mathscr{P}$ be defined by: "$[u,v]$ contains all large enough integers". This property is obviously global. The assertion "$[u,v] \in \mathscr{P}$" means that v is greater than all the integers, and that u is not greater than all the integers; thus $\mathscr{P}$ is subtractive (see Example 3). By the GL axiom, $\mathscr{P}$ has a limit point x, i.e., $k \to x$. If x were finite, we could find an interval of the type $[v-1,v]$ (where v is finite) which contains all integers $\geq k$. For such a k, we have $v-1 < k \implies v < k+1$, which is impossible. Therefore, $x = +\infty$. □

An alternate proof uses the GL axiom and the property defined by "$[u,v]$ contains infinitely many integers".

4.6 Cauchy Sequences

Instead of the classical definition of a Cauchy sequence using double indexing, we feel that the following definition is somehow simpler.

Definition 16. A sequence (x_k) is *Cauchy* if for all $\varepsilon > 0$, we have, starting from a certain index K, the inequality $|x_K - x_k| < \varepsilon$ for all $k \geq K$.

In other words, the sequence is eventually "almost stationary". This definition, of course, requires that the x_k are eventually all finite. The usual definition of the usual convergence (towards a finite limit x) replaces x_K by x in the above inequality. Since $|x_K - x_k| \leq |x_K - x| + |x - x_k|$, any convergent sequence is a Cauchy sequence.

Remark 1. The classical definition is: for all $\varepsilon > 0$, we have $|x_\ell - x_k| < \varepsilon$ for all large enough k and ℓ ($\geq K$). Since $|x_\ell - x_k| \leq |x_\ell - x_K| + |x_K - x_k|$ this is equivalent to the above definition.

Remark 2. A Cauchy sequence is bounded, because for any given $\varepsilon > 0$, we have $|x_k| < \varepsilon + |x_K|$ for all $k \geq K$.

The following *Cauchy criterion* is Bolzano's theorem:

Theorem 6 (Cauchy Cauchy's Completeness). *A sequence is convergent if and only if it is a Cauchy sequence.*

Proof. It is enough to show that a Cauchy sequence (x_k) converges. Property $\mathscr{P}$ defined by "$[u,v]$ contains every x_k for all large enough k". is obviously global. It is also o-subtractive, because if $t < u < v < w$ and $[t,w]$ satisfies $\mathscr{P}$, it is impossible for both $[t,u]$ and $[v,w]$ to contain x_k for infinitely many values of k, since that would contradict the Cauchy property $|x_\ell - x_k| < \varepsilon$ for $\varepsilon = (v-u)$. Hence either $[t,v] \in \mathscr{P}$ or $[u,w] \in \mathscr{P}$. By the GL axiom, $\mathscr{P}$ has a limit point x, that is, $x_k \to x$. The limit x is finite because the sequence (x_k) is bounded. □

4.7 Nested Intervals of Cantor and Adjacent Sequences

We consider families of intervals $[r,s]$ belonging to $[a,b]$. Such families do not have a common point if their intersection is empty.

Theorem 7 (Cantor). *Each family of intervals $[r,s]$ with no common point admits a finite subfamily with no common point.*

Proof. Let $\mathscr{S}$ be the family of intervals $[r,s]$ with no common point:

$$\bigcap_{[r,s]\in\mathscr{S}} [r,s] = \emptyset.$$

Let $\mathscr{P}$ be the property that there is a finite sub-family of $\mathscr{S}$ with no common point in $[u,v]$. This property is local. Indeed, no point x belongs to all intervals of $\mathscr{S}$, so there exists an interval $[r,s] \in \mathscr{S}$ which does not contain x. It is possible to chose a neighborhood $V(x)$ disjoint from this interval $[r,s]$. Each $[u,v]$ adapted to $V(x)$ then satisfies $\mathscr{P}$, since it does not contain any point of $[r,s]$ (which by itself constitutes a finite sub-family of $\mathscr{S}$). The property $\mathscr{P}$ is also additive, because given $u < v < w$, if $[u,v]$ does not contain a common point of a finite sub-family of $\mathscr{S}$, and if $[v,w]$ does not contain a common point of another finite sub-family of $\mathscr{S}$, then $[u,v]$ does not contain a common point of the union of the two finite sub-families. By the LG axiom, $[a,b] \in \mathscr{P}$, that is, there exists a finite sub-family of $\mathscr{S}$ without any common point in $[a,b]$. □

Theorem 8 (Nest: Cantor's Property). *Every sequence $[r_k, s_k]$ of nested intervals (such that $[r_{k+1}, s_{k+1}] \subset [r_k, s_k]$ for all k) has a common point (common to each of the intervals).*

This is an immediate consequence of Cantor's theorem, since all finite sub-families of nested intervals sequences $[r_k, s_k]$ has as an intersection in the last (smallest) interval, which is not-empty. As a consequence the sequence has a common point. However it is instructive to show a direct proof using the GL axiom:

Proof. Let $\mathscr{P}$ be the property that $[u, v]$ contains one of the intervals $[r_k, s_k]$ (hence all intervals for large enough k). The property $\mathscr{P}$ is clearly global. If it is *not* subtractive, there exists $u < v < w$ such that $[u, w] \in \mathscr{P}$ with $r_k < v < s_k$ for all k large enough: v is then a common point for all such intervals. Otherwise, $\mathscr{P}$ is subtractive, and by the GL axiom, $\mathscr{P}$ has a limit point x. This point is necessarily common to the intervals $[r_k, s_k]$, otherwise we could find an interval $[u, v]$ which contains x and disjoint from an interval $[r_k, s_k]$, which contradicts that $\mathscr{P}$ has a limit at x. □

An easy consequence is

Theorem 9 (Cantor). $[a, b]$ *is uncountable.*

Proof. We may assume that $[a, b]$ is bounded. If it is countable, let us write $[a, b] = \{x_1, x_2, \ldots, x_k, \ldots\}$. Set $[a_0, b_0] = [a, b]$. For all integer $k \geq 0$, define by induction a subinterval $[a_{k+1}, b_{k+1}]$ of $[a_k, b_k]$ which does not contain x_k (by example, partition $[a_k, b_k]$ into three sub-intervals of the same length, and define $[a_{k+1}, b_{k+1}]$ as the first sub-interval of three that does not contain x_k). The sequence of intervals $[a_k, b_k]$ has a common point $x \in [a, b]$ distinct from all of the $x_1, x_2, \ldots, x_k, \ldots$, which is impossible. □

Definition 17. Two sequences (r_k), (s_k) in $[a, b]$ are *adjacent* if (r_k) is nondecreasing, (s_k) is nonincreasing, and $s_k - r_k$ tends to 0.

Theorem 10 (Adjacent Sequences). *Two adjacent sequences converge to the same limit.*

Proof. The difference $s_k - r_k$ decreases since r_k increases and s_k decreases. As it tends to zero, it is always ≥ 0. So $r_k \leq s_k$ for every k, and the intervals $[r_k, s_k]$ are nested. Let x be a common point in these intervals: $r_k \leq x \leq s_k$ for every k. Since the width of $[r_k, s_k]$ tends to zero, each neighborhood $V(x)$ contains $[r_k, s_k]$ for large enough k. Hence both sequences tend to x. □

For completeness we observe the following.

Theorem 11. *The LG axiom is equivalent to the two theorems of the adjacent sequences 10 and of Archimedes 5.*

Proof. It is enough to show that both theorems imply the GL axiom. Let $\mathscr{P}$ be global and subtractive. One proceeds by dichotomy. Let $[a_0, b_0] = [a, b]$ unless $[a, b] = [-\infty, +\infty]$, in which case we can define $[a_0, b_0]$ to be equal to the first of the two intervals $[-\infty, 0]$ or $[0, +\infty]$ which satisfies $\mathscr{P}$. Thus we can always assume that at least one of the two interval extremities are finite. We define them by induction $[a_{k+1}, b_{k+1}]$ equal the first of the two following intervals that satisfy $\mathscr{P}$:

- $[a_k,(a_k+b_k)/2]$ or $[(a_k+b_k)/2,b_k]$, if a_k and b_k are finite;
- $[a_k,a_k+1]$ or $[a_k+1,+\infty]$, if $b_k=+\infty$ (a_k being finite);
- $[-\infty,b_k-1]$ or $[b_k-1,b_k]$, $a_k=-\infty$ (b_k being finite).

As $\mathscr{P}$ is subtractive, this sequence is well defined. Since (a_k) is nondecreasing, and (b_k) is nonincreasing, we have three cases to consider:

- a_k and b_k are finite for large enough k; then $b_k-a_k=2^{-k}(b-a)$ tends to 0 since (by Theorem 5) $2^k>k\to+\infty$; the sequences (a_k) and (b_k) are adjacent, so by Theorem 10 they converge to the same limit.
- $b_k=+\infty$ for every k; we then have $a_0=0$ and $a_{k+1}=a_k+1$ for any $k\geq 0$, where $a_k=k\to\infty$ by Theorem 5.
- $a_k=-\infty$ for every k; we then have the same $b_k=-k\to-\infty$ by Theorem 5.

In all cases, a_k and b_k tend to the same limit x (finite or infinite). Each neighborhood $V(x)$ for every k large enough, contains the interval $[a_k,b_k]\in\mathscr{P}$. Therefore $\mathscr{P}$ has a limit point x. ☐

4.8 Bolzano-Weierstrass Property

Definition 18. A *limit point* x of a set E (also called *accumulation point*) is such that each neighborhood $V(x)$ contains infinitely many points of E. That is to say, the property that $[u,v]$ contains infinitely many points of E has x as a limit point in the sense of definition 11.

A *limit point* x of a sequence (x_k) (also called *cluster point*, or *adherent value*) is such that each neighborhood $V(x)$ contains x_k for infinitely many values of k. That is to say, the property that $[u,v]$ contains x_k for infinitely many values of k has x as a limit point in the sense of definition 11.

Theorem 12 (BW: Bolzano-Weierstrass). *Any* infinite *set of points has a limit point.*

If the set is bounded, this limit point is finite.

Proof. Let E be an infinite set of points, and $\mathscr{P}$ be the property with the interval $[u,v]$ containing infinitely many points in E. This property is global by hypothesis. It is also subtractive: if $[u,w]$ contains infinitely many points in E, at least one of those sub-intervals $[u,v]$, $[v,w]$ must have infinitely many points in E. By the GL axiom, $\mathscr{P}$ has a limit point x, i.e., x is a limit point of E. ☐

Note that by contraposition, any locally finite set is finite. Also, by the same argument, any uncountable set of points has a *condensation point* (that is, such that every neighborhood of it contains uncountably many points of E).

Theorem 13 (BW for Sequences). *Any sequence has a limit point.*

This is a consequence of the preceding theorem applied to the set of the sequence values, if one considers the two cases where the set is finite or infinite. A direct proof using the GL axiom is as follows.

Proof. Let (x_k) be any point sequence and let $\mathscr{P}$ be the property that an interval $[u,v]$ contains x_k for infinitely many values of k. The property is evidently global. It is also subtractive: if $[u,w]$ contains x_k for infinitely many values of k, at least one of $[u,v]$, $[v,w]$ has the same property. By the GL axiom, $\mathscr{P}$ has a limit point x, that is, x is a limit point (in the usual sense of definition 18) of the sequence (x_k). □

4.9 Heine-Borel-Lebesgue Covering

Recall that a family of intervals *cover* a set of points E if each point in E is in at least one of the intervals of that family. For example, $\mathscr{R} = \{]r_i,s_i[\}_{i\in I}$ covers E if $E \subset \bigcup_{i\in I}]r_i,s_i[$. We also say that this family is a *cover* of E. In addition, if this family is composed of a finite number of intervals, then it is a *finite cover* of E.

Theorem 14 (BL: Borel-Lebesgue (sometimes known as Heine-Borel)**).** *Any cover of $[a,b]$ by means of open intervals admits a finite subcover.*

In other words, given any family $\mathscr{R}$ of open intervals covering $[a,b]$, we can find a finite number of intervals $]r_k,s_k[$ $(k=1,\dots,m)$ of $\mathscr{R}$ such that each point in $[a,b]$ belongs to at least one of $]r_k,s_k[$:

$$[a,b] \subset \bigcup_{k=1}^{m}]r_k,s_k[.$$

Proof. Let $\mathscr{P}$ be the property that the interval $[u,v]$ is covered by a *finite* number of intervals $]r,s[$ of $\mathscr{R}$. As $\mathscr{R}$ covers $[a,b]$, each $x \in [a,b]$ belongs to some interval $]r,s[$ of $\mathscr{R}$. Taking $V(x) =]r,s[$ as a neighborhood of x, every interval $[u,v] \subset V(x)$ is covered by a finite number (equal to 1) of intervals of $\mathscr{R}$, that is, $]r,s[$ itself. Hence $\mathscr{P}$ is local. It is also additive: if $[u,v]$ and $[v,w]$ covered each one by a finite number of open intervals of $\mathscr{R}$, their union is a finite cover of $[u,w]$. By the LG axiom, $\mathscr{P}$ is global, i.e., satisfied by $[a,b]$. □

4.10 Cousin Partition

Definition 19. A *partition* (of intervals) of $[a,b]$ is a finite cover of $[a,b]$ by non-overlapping intervals $[u,v] \subset [a,b]$.

In other words, a partition of $[a,b]$ corresponds to a subdivision, that is, a finite number of points $a = u_1 < u_2 < \ldots < u_m = b$, such that the intervals $[u_k,u_{k+1}]$ $(1 \le i < m)$ do not overlap and cover $[a,b]$.

Definition 20. An *environment* V of a set of points E is a family of neighborhoods, with one neighborhood $V(x)$ for every point $x \in E$.

Definition 21. An interval $[u,v]$ is *adapted* to the environment V if there exists $x \in [u,v]$ such that $[u,v] \subset V(x)$. A set of intervals is *adapted* to the environment V if every interval is.

Thus, a partition $\pi = \{[u_i, u_{i+1}]\}_{1 \leq i < m}$ of $[a,b]$ is *adapted* to the environment V (on $[a,b]$) if it exists for each of the intervals of which it is composed, that is, if for all i ($1 \leq i < m$), there exists $x \in [u_i, u_{i+1}] \subset V(x)$.

Theorem 15 (Cousin: Cousin's Partition). *For any environment V of $[a,b]$, there exists a partition of $[a,b]$ adapted to V.*

Proof. Let $\mathscr{P}$ be the property that the interval $[u,v]$ admits a partition adapted to V. For all $x \in [a,b]$, every adapted interval $[u,v]$ to $V(x)$ is for itself an adapted partition of $[u,v]$ to V. Hence $\mathscr{P}$ is local. It is also additive: if $[u,v]$ and $[v,w]$ admit each one a partition adapted to V, the union of the two partitions constitute a partition of $[u,w]$ adapted to V. By the LG axiom, $\mathscr{P}$ is global. □

For completeness we observe the following.

Proposition 8. *The LG axiom is equivalent to Cousin's theorem.*

Proof. It is enough to show that Cousin's theorem implies the LG axiom. Let $\mathscr{P}$ be a local and additive property in $[a,b]$. As $\mathscr{P}$ is local, there exists an environment V such that every adapted interval to V satisfies $\mathscr{P}$. A Cousin's partition corresponding to V is then composed of intervals that satisfy $\mathscr{P}$. Since $\mathscr{P}$ is additive, it follows that $[a,b] \in \mathscr{P}$: $\mathscr{P}$ is global. □

5 Equivalence Between Completeness Axioms

In this section we prove the equivalence between the various completeness axioms. This of course is not required in an elementary course but is satisfactory for logical consistency.

Proposition 9. *Local-global (LG or GL) axioms are equivalent to any of the following statements:*

1. *Existence of a Dedekind cut point;*
2. *Existence of supremum (or infimum);*
3. *Principle of continuous induction;*
4. *Intersection theorems of Cantor + Archimedean property;*
5. *Nested intervals theorem + Archimedean property;*
6. *Adjacent sequences theorem + Archimedean property;*
7. *Monotone limit theorem;*

8. *Cauchy's criterion + Archimedean property;*
9. *Bolzano-Weierstrass theorem (for sets or for sequences);*
10. *Heine-Borel-Lebesgue covering theorem;*
11. *Existence of Cousin's partition.*

Proof. We have already directly proven each one of the results from LG or GL axioms. The converse proofs given above show the equivalences with the statements 1, 2, 3, 6, and 11. Furthermore, the implications $4 \implies 5 \implies 6$ have already been seen, hence the equivalences with statements 4 and 5.

We can conclude with the following implications: ($9 \implies 7 \implies 6$), ($8 \implies 6$), and ($10 \implies 11$).

$9 \implies 7$: We have seen that the Bolzano-Weierstrass theorem for sets implies those for sequences. We now show that the Bolzano-Weierstrass theorem for sequences implies the monotone convergence theorem. Let (x_k) be a monotonic sequence and $V(x)$ an interval which is a neighborhood of x. The sequence (x_k) admits a limit point x such that $x_\ell \in V(x)$ for infinitely many values of ℓ. Let K be such that $x_K \in V(x)$ and $k \geq K$. There exists an index $\ell > k$ such that $x_\ell \in V(x)$. As this sequence is monotonic, x_k lies between x_K and x_ℓ, hence $x_k \in V(x)$ for all $k \geq K$, which shows that x is the limit of (x_k).

$7 \implies 6$: If two adjacent sequences (r_k), (s_k) are monotone, then they tend to limits: $r_k \to r$ and $s_k \to s$. Since $s_k - r_k \to 0$, we deduce $s - r = 0$, hence $r = s$, which proves the adjacent sequences theorem.
The sequence $x_k = k$ is increasing and tends to a limit x. If x is finite, we have both $k \to x$ and $k+1 \to x+1$ so $x = x+1$ which is impossible. Hence $x = +\infty$, which proves the Archimedes property.

$8 \implies 6$: It is enough to show that the Cauchy criterion implies the adjacent sequences theorem. Two adjacent sequences (r_k), (s_k) are such that $s_k - r_k$ is nonincreasing and tends to 0. For all $\varepsilon > 0$, we have then $s_K - s_k \leq s_K - r_K < \varepsilon$ and $r_k - r_K \leq s_K - r_K < \varepsilon$ for all $k \geq K$. These are Cauchy sequences, hence converge: $r_k \to r$ and $s_k \to s$. As $s_k - r_k \to 0$, we deduce that $s - r = 0$ or $r = s$.

$10 \implies 11$: Let V be an environment of $[a,b]$ which we assume is of open intervals. This constitutes a covering of $[a,b]$. Extract a finite covering $V(x_k) =]r_k, s_k[$ for $1 \leq k \leq K$. Relabelling if necessary, we can always assume that $x_1 < x_2 < \cdots < x_K$ and assume that K is minimal. Pick x_{i_1} such that $a \in V(x_{i_1})$ and as long as $b \notin V(x_{i_j})$ define a finite sequence x_{i_j} such that $x_{i_{j+1}} > x_{i_j}$ and $V(x_{i_{j+1}})$ overlaps with $V(x_{i_j})$, until $V(x_{i_{m-1}}) \ni b$. One obtains a finite covering $V(x_{i_k}) =]r_{i_k}, s_{i_k}[$ for $1 \leq k < m$ where each $V(x_{i_k})$ overlaps with $V(x_{i_{k+1}})$. We can then choose m points $a = u_1 < u_2 < \ldots < u_m = b$ with $u_{k+1} \in V(x_{i_k}) \cap V(x_{i_{k+1}})$. Every $[u_k, u_{k+1}]$ is then adapted to $V(x_{i_k})$ for $1 \leq k < m$. These intervals forms an adapted Cousin partition to V. □

6 Elementary Theorems of Real Analysis

6.1 Continuous Functions

We consider functions defined on $[a,b]$, with values in $\mathbb{R}$ or $\mathbb{C}$, or more generally in $\mathbb{R}^n$ or $\mathbb{C}^n$, or every vector space of finite dimension over $\mathbb{R}$ or $\mathbb{C}$, or even more generally, in a Banach space over $\mathbb{R}$ or $\mathbb{C}$. We note $|\cdot|$ the corresponding absolute value, modulus, or norm.

We advocate the following definition of continuity.

Definition 22. A function f is *continuous* in a point x if for all $\varepsilon > 0$, the property "$|f(v) - f(u)| < \varepsilon$" is local in x. A function f is *continuous* on a set E if it is continuous in each point of E.

In other words, f is continuous at x if for all $\varepsilon > 0$, there exists a neighborhood $V(x)$ in which $u \leq x \leq v$ implies $|f(u) - f(v)| < \varepsilon$. When $x = b$, it is a continuity to the left at b; when $x = a$, it is a continuity to the right at a.

Conversely, f is discontinuous at x if there exists $\varepsilon > 0$ for which the property "$|f(v) - f(u)| \geq \varepsilon$" is not limit at x.

Remark 3. The continuity definition in a point x is equivalent to the classical definition: for all $\varepsilon > 0$, there exists a neighborhood $V(x)$ such that for all $t \in V(x)$, $|f(t) - f(x)| < \varepsilon$. Indeed we obtain this condition from the definition above when getting $[u,v] = [x,t]$ if $t \geq x$, $[u,v] = [t,x]$ otherwise. Conversely, if $[u,v]$ is adapted to $V(x)$, we have $|f(v) - f(u)| \leq |f(v) - f(x)| + |f(u) - f(x)| < 2\varepsilon$.

Remark 4. Saying that f is continuous at x is the same as saying that $f(t)$ tends toward $f(x)$ when $t \to x$. When $x = b$, this is a limit to the left at b, denoted $f(b^-)$; when $x = a$, this is a limit to the right at a, denoted $f(a^+)$.

It is possible for f to only be defined on $]a,b[$ but having finite limits $f(a^+)$ and $f(b^-)$. We can say then that f is *continuous* on $[a,b]$, in the sense where we can extend by continuity f by setting $f(a) = f(a^+)$ and $f(b) = f(b^-)$.

Example 12. The function $f(x) = \dfrac{1}{1+x^2}$ is continuous on $[-\infty, +\infty]$ (extend by continuity setting $f(\pm\infty) = 0$).

Example 13. The function $f(x) = \arctan(x)$ is continuous on $[-\infty, +\infty]$ (extend by continuity setting $f(-\infty) = -\pi/2$ and $f(+\infty) = \pi/2$).

Example 14. The function $f(x) = x^2$ is continuous on any bounded interval. But with our definition[7] it is not continuous (cannot be extended by continuity) on $[-\infty, +\infty]$.

Theorem 16 (BT: Boundedness Theorem). *Each continuous function on $[a,b]$ is bounded on $[a,b]$.*

[7] We found it convenient for later developments that functions assume only finite values in order to leverage on the complete metric space property of the set of function values.

Notice that $[a,b]$ may very well be unbounded (a and/or b can be infinite).

Proof. The property $\mathscr{P}$ defined by "f is bounded on $[u,v]$" is local since for given $\varepsilon > 0$, any u in the neighborhood $V(x)$ satisfies $|f(u)| \leq |f(u) - f(x)| + |f(x)| \leq |f(x)| + \varepsilon$. The property $\mathscr{P}$ is also additive, because if $|f| \leq M$ on $[u,v]$ and $|f| \leq M'$ on $[v,w]$ then $|f| \leq \max(M, M')$ on $[u,w]$. By the LG axiom, f is bounded on $[a,b]$. □

The above proof shows that, more generally, any *locally bounded* function is (globally) bounded.

Theorem 17 (EVT: Extreme Value Theorem). *Each continuous real function f on $[a,b]$ reaches its maximum, i.e., there exists $x \in [a,b]$ such that $f(x) \geq f(t)$ for all $t \in [a,b]$.*

Of course, considering $-f$, each real continuous function f on $[a,b]$ reaches its minimum.

Proof. The property $\mathscr{P}$: "there does not exist a value of f which is greater than any value of f on $[u,v]$" (in other words $\forall t \in [a,b],\ \exists x \in [u,v],\ f(t) \leq f(x)$) is clearly global. It is also subtractive, otherwise the greater of the two values of f is greater than f on $[u,w]$. By the GL axiom, $\mathscr{P}$ has a limit point x. We have then, by continuity, $\forall t \in [a,b], f(t) < f(x) + \varepsilon$ for all $\varepsilon > 0$, hence $f(t) \leq f(x)$ for all $t \in [a,b]$. □

Interestingly, the proof extends verbatim to the more general case where f is *upper semi-continuous*. Also notice that the boundedness theorem was not required for this proof, which, therefore, provides another proof for boundedness since $\min f \leq f \leq \max f$.

Theorem 18 (IVT: Intermediate Value Theorem). *Each continuous real function on $[a,b]$ takes any value between $f(a)$ and $f(b)$.*

In other words, if y is any value taken between $f(a)$ and $f(b)$, there exists x such that $f(x) = y$.

Proof. The property $[u,v] \in \mathscr{P}$ defined by "y is between $f(u)$ and $f(v)$" (that is, "$f(u) \leq y \leq f(v)$ or $f(v) \leq y \leq f(u)$") is global. It is also subtractive, because if $u < v < w$ and y is between $f(u)$ and $f(w)$, then whatever the value of $f(v)$, y is either between $f(u)$ and $f(v)$, or between $f(v)$ and $f(w)$. By the GL axiom, $\mathscr{P}$ has a limit point x. By continuity, for all $\varepsilon > 0$, we have $f(x) - \varepsilon < y < f(x) + \varepsilon$, hence $y = f(x)$. □

Theorem 19 (Heine's Theorem). *Every continuous function of a bounded interval $[a,b]$ is uniformly continuous.*

Proof. Let f be continuous on $[a,b]$ and $\varepsilon > 0$. The property $\mathscr{P}$: "$|f(x) - f(y)| < \varepsilon$ for all x, y sufficiently close in $[u,v]$" (that is, there exists $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$) is local by definition of continuity. It is also o-additive:

if $t < u < v < w$ and if for all x, y sufficiently close in $[t, v]$ or in $[u, w]$ one has $|f(x) - f(y)| < \varepsilon$, then all $x, y \in [t, w]$ such that $|x - y| < v - u$ will be both in $[t, v]$ or both in $[u, w]$, so that we will always have $|f(x) - f(y)| < \varepsilon$ for sufficiently close x, y. By the LG axiom, $\mathscr{P}$ is global, which means that f is uniformly continuous on $[a, b]$. □

6.2 Differentiable Functions

We advocate the following definition of differentiability.

Definition 23. A function f is *differentiable* at a finite point x with derivative $f'(x) = \lambda$ if for all $\varepsilon > 0$, the property $[u, v] \in \mathscr{P}$ defined by

$$|f(v) - f(u) - \lambda(v - u)| < \varepsilon(v - u)$$

is local in x. A function f is *differentiable* on a (bounded) set E if it is differentiable at all points of E which defines the *derivative* f' of f on E.

In other words, f has a derivative $\lambda = f'(x)$ at x if for all $\varepsilon > 0$, there exists a neighborhood $V(x)$ in which $u \leq x \leq v$ implies $|f(v) - f(u) - f'(x) \cdot (v - u)| < \varepsilon(v - u)$. If f is defined on $[a, b]$ and a or b are finite, we can have $x = a$ or $x = b$ in the definition of the derivative; this is then a derivative to the left at b if $x = b$, a derivative to the right at a if $x = a$.

Notice that if f takes values in a vector space, $\lambda \cdot (v - u)$ is the vector product λ by the scalar $(v - u)$, and the derivative $f'(x)$ has vector values.

Remark 5. For real-valued functions the above definition of derivative at x point is equivalent to the classical definition:

$$\frac{f(x) - f(t)}{x - t} \to \lambda$$

where t tends to x. Indeed, we get this condition from the definition above by taking $[u, v] = [x, t]$ if $t \geq x$, $[u, v] = [t, x]$ otherwise, and dividing by $(v - u)$. Conversely, if $[u, v]$ is adapted to $V(x)$, we have $|f(v) - f(u) - \lambda(v - u)| = |f(v) - f(x) - \lambda(v - x) + f(x) - f(u) - \lambda(x - u)| \leq \varepsilon\big(|v - x| + |x - u|\big) = \varepsilon(v - u)$. (This is sometimes referred to in the literature as the "straddle lemma").

Proposition 10. *Every differentiable function is continuous.*

Proof. If f is differentiable with derivative λ at x, the property $|f(v) - f(u)| < (|\lambda| + \varepsilon)(v - u)$ is local at x. But if $[u, v]$ is small enough, for a given $\varepsilon' > 0$, $(|\lambda| + \varepsilon)(v - u) < \varepsilon'$, hence the property $|f(v) - f(u)| < \varepsilon'$ is local at x. □

Inspired by Cohen and Bers [3, 6], we advocate the use of the following theorem in place of the classical mean value theorem (see Remark 6 below).

Theorem 20 (Finite Increase Inequality[8]). *Let f,g be two differentiable functions on $[a,b]$, where g assumes real values.*

- *if $|f'| < g'$ on $[a,b]$ then $|f(b)-f(a)| < g(b)-g(a)$;*
- *if $|f'| \leq g'$ on $[a,b]$ then $|f(b)-f(a)| \leq g(b)-g(a)$;*

Proof. Let $x \in [a,b]$, suppose $|f'(x)| < g(x)$ and let $\varepsilon > 0$ small enough such that $|f'(x)| + \varepsilon < g'(x) - \varepsilon$.

For $u \leq x \leq v$ in a neighborhood $V(x)$, one has $|f(v)-f(u)| < |f'x)| \cdot (v-u) + \varepsilon(v-u) < g'(x) \cdot (v-u) - \varepsilon(v-u) < g(v)-g(u)$. This implies that the property "$|f(v)-f(u)| < g(v)-g(u)$" is local. This property is also additive for if $u < v < w$, $|f(v)-f(u)| < g(v)-g(u)$ and $|f(w)-f(v)| < g(w)-g(v)$ imply $|f(w)-f(u)| \leq |f(w)-f(v)| + |f(v)-f(u)| < g(w)-g(v)+g(v)-g(u) = g(w)-g(u)$. By the LG axiom, the property is global: $\big|f(b)-f(a)\big| < g(b)-g(a)$.

For the second part, we can replace $g(x)$ by $g(x)+\varepsilon x$ for $\varepsilon > 0$, so that $|f'| < g'+\varepsilon$, hence $|f(b)-f(a)| < g(b)-g(a)+\varepsilon(b-a)$. Since $\varepsilon > 0$ is arbitrarily small, we have $|f(b)-f(a)| \leq g(b)-g(a)$. □

An alternate proof for the second part considers the property "$|f(v)-f(u)| \leq g(v) - g(u) + \varepsilon(v-u)$".

The above theorem is enough to prove the following important results:

- if $f' = 0$ on $[a,b]$ then f is constant there (take g constant); thus an antiderivative is unique up to an additive constant;
- if $g' > 0$ on $[a,b]$ then g is increasing; if $g' \geq 0$ on $[a,b]$ then g is nondecreasing (take f constant).

Remark 6. Cohen and Bers wrote:

> *With characteristic vigor, L. Bers announced in a recent conversation: "Who needs the mean value theorem! All we want as a start in elementary calculus is the proposition that if $f'(x) = 0$ for all x in $[a,b]$, then f is constant." [6]*
>
> *The "full" mean value theorem [...] is a curiosity. It may be discussed together with another curiosity, Darboux' theorem that every derivative obeys the intermediate value theorem. [3]*

The actual "mean value theorem", which states that there exists $x \in \,]a,b[$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(x).$$

gives no indication of the position of point x in the interval $[a,b]$. This proof shows that the only condition is that $\frac{f(b)-f(a)}{b-a}$ lies between two possible values of the derivative, and as such, is related to Darboux's theorem that f' takes any value between $f'(a)$ and $f'(b)$ on $[a,b]$. We couldn't find a direct and easy proof of the mean value theorem using the LG or GL axiom. Darboux's theorem can be proved using the fact that the derivative f' vanishes at an extremum of f, and the mean value

[8] A literal translation of the French "inégalité des accroissements finis", which advantageous replaces the "théorème des accroissements finis", which is the mean value theorem.

theorem then becomes an easy consequence of Darboux's theorem: if f' does not take the value $\lambda = \frac{f(b)-f(a)}{b-a}$ then by Darboux's theorem f' is either always greater or always less, which by the finite increase inequality implies either $\frac{f(b)-f(a)}{b-a} > \lambda$ or $< \lambda$, a contradiction.

7 Conclusion and Perspectives

Our objective is twofold. First we would like to draw attention to the local-global principle as a new efficient and enjoyable tool for proving the basic theorems of real analysis. Second, we aim to clarify the local-global concept to possibly improve the teaching of real analysis at undergraduate and graduate levels.

As a future work the LG/GL concept may be used as a basis for a new presentation of the integral, just as Cousin's lemma was used to build the Kurzweil-Henstock integral [13, 18, 23]. In such an approach the so-called "fundamental theorem of calculus", appropriately generalized, can become the actual definition for a novel notion of the antiderivative function.

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