Weakly stationary stochastic processes valued in a separable Hilbert space: Gramian-Cramér representations and applications

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Abstract

The spectral theory for weakly stationary processes valued in a separable Hilbert space has known renewed interest in the past decade. However, the recent literature on this topic is often based on restrictive assumptions or lacks important insights. In this paper, we follow earlier approaches which fully exploit the *normal Hilbert module* property of the space of Hilbert-valued random variables. This approach clarifies and completes the isomorphic relationship between the *modular spectral domain* to the *modular time domain* provided by the Gramian-Cramér representation. We also discuss the general Bochner theorem and provide useful results on the composition and inversion of lag-invariant linear filters. Finally, we derive the Cramér-Karhunen-Loève decomposition and harmonic functional principal component analysis without relying on simplifying assumptions.

1 Introduction

Functional data analysis has become an active field of research in the recent decades due to technological advances which makes it possible to store longitudinal data at very high frequency (see e.g. [22, 31]), or complex data e.g. in medical imaging [18, Chapter 9], [15], linguistics [28] or biophysics [27]. In these frameworks, the data is seen as valued in an infinite dimensional separable Hilbert space thus isomorphic to, and often taken to be, the function space $L^2(0,1)$ of Lebesgue-square-integrable functions on $[0,1]$. In this setting, a functional time series refers to a bi-sequences $(X_t)_{t\in\mathbb{Z}}$ of $L^2(0,1)$-valued random variables and the assumption of finite second moment means that each random variable X_t belongs to the L^2 Bochner space $L^2(\Omega,\mathcal{F},L^2(0,1),\mathbb{P})$ of measurable mappings $V:\Omega\to L^2(0,1)$ such that $\mathbb{E}\left[\|V\|^2_{L^2(0,1)}\right]<\infty$, where $\|\cdot\|_{L^2(0,1)}$ here denotes the norm endowing the Hilbert space $L^2(0,1)$.

Spectral analysis of weakly stationary functional time series has been recently considered in [19, 20, 26] where, in particular, the authors derive a functional version of the Cramér representation. The functional Cramér representation of [19, 20, 26] relies on a spectral density operator defined under strong assumptions on the covariance structure of the time series. Under the same assumption, [20] introduced filters whose transfer functions are valued in a restricted set of operators and this was latter generalized to bounded-operator-valued transfer functions in [26, Section 2.5] (see also [29, Appendix B.2.3]). A more general approach is adopted in [30] where the authors provide a definition of operator-valued measures from which they derive a functional version of the Herglotz theorem, the functional Cramér representation and the definition of linear filters with bounded-operator-valued transfer functions.

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On the other hand, to a certain extent, a general spectral theory for weakly stationary time series has been originally developed in a very general fashion much earlier, starting from the seminal works by Kolmogoroff, [14], and spanning over several decades, see [11] and the references therein. These foundations include time domain and frequency domain analyses, Cramér (or spectral) representations, the Herglotz theorem and linear filters. In [14, 11] the adopted framework is that of a bi-sequence $X = (X_t)_{t\in\mathbb{Z}} \in \mathcal{H}^{\mathbb{Z}}$ valued in a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ and weakly stationary in the sense that $\langle X_s, X_t\rangle_{\mathcal{H}}$ only depends on the lag $s-t$. Taking $\mathcal{H}$ to be the L^2 Bochner space $L^2(\Omega, \mathcal{F}, L^2(0,1), \mathbb{P})$, this framework directly applies to functional time series. At first sight, it is thus fair to question the novelty of the theory developed in [19, 20, 26, 29, 30] in comparison to results seemingly already available in a very general fashion in the original works that founded the modern theory of stochastic processes. This novelty issue cannot be unequivocally answered because there are (many) different approaches to establishing a spectral theory for weakly stationary functional time series. Moreover, the merits and the drawbacks of a specific approach depend on the applications that one wishes to deduce from the spectral theory at hand and on the required mathematical tools in which one is ready to invest in order to rigorously employ it.

In the framework of [11] recalled above, a linear filter is a linear operator on H_X onto H_X which commutes with the lag operator U^X, where H_X is the closure in $\mathcal{H}$ of the linear span of $(X_t)_{t\in\mathbb{Z}}$ and U^X is the operator defined on H_X by mapping X_t to X_{t+1} for all $t \in \mathbb{Z}$. As explained in [11, Section 3], a complete description of such a filter is given in the spectral domain by its transfer function. Let us recall the essential formulas which summarize what this means. In [11], the spectral theory follows from and start with the *canonical representation* of the lag operator U^X above, namely

$$U^X = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda}\, \xi(\mathrm{d}\lambda) , \tag{1.1}$$

where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and ξ is the spectral measure of U^X (which is a measure valued in the space of operators on H_X onto itself). This corresponds to [11, Eq. (8)] with a slightly different notation. Then defining $\hat{X}$ as $\xi(\cdot)X_0$ (thus a measure valued in H_X), one gets the celebrated Cramér representation (see [11, Eq. (13a)] again with a slightly different notation)

$$X_t = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda\, t}\, \hat{X}(\mathrm{d}\lambda) , \quad t \in \mathbb{Z} . \tag{1.2}$$

An other consequence of (1.1) is what is called the Herglotz theorem in [11, Eq. (9)], summarized by the formula

$$\langle X_s, X_t\rangle_{\mathcal{H}} = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda\,(s-t)}\, \mu(\mathrm{d}\lambda) , \quad s,t \in \mathbb{Z} , \tag{1.3}$$

where $\mu = \langle \xi(\cdot)X_0, X_0\rangle_{\mathcal{H}}$ is a non-negative measure on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$. Interpreting the right-hand side of (1.3) as the scalar product of the two functions $\mathrm{e}_s : \lambda \mapsto \mathrm{e}^{\mathrm{i}\lambda s}$ and $\mathrm{e}_t : \lambda \mapsto \mathrm{e}^{\mathrm{i}\lambda t}$ in $L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu)$, Relation (1.3) is simply saying that the Cramér representation (1.2) mapping e_t to X_t is isometric. Following this interpretation, one can extend this isometric mapping to a unitary operator between the two isomorphic Hilbert spaces $L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu)$ and H_X, respectively refered to as the *spectral domain* and the *time domain*. In particular the output of a linear filter with transfer function $\Phi \in L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu)$ is given by

$$Y_t = \int \mathrm{e}^{\mathrm{i}\lambda t}\, \Phi(\lambda)\, \hat{X}(\mathrm{d}\lambda) , \quad t \in \mathbb{Z} , \tag{1.4}$$

or in other words, Y_t is the image of the function $\mathrm{e}_t\Phi$ by the extended unitary operator that maps the spectral domain to the time domain.

The spectral theory (1.1)–(1.4) applies to multivariate time series by taking $\mathcal{H} = L^2(\Omega, \mathcal{F}, \mathbb{C}^q, \mathbb{P}) = (L^2(\Omega, \mathcal{F}, \mathbb{P}))^q$ and to functional time series by taking $\mathcal{H} = L^2(\Omega, \mathcal{F}, L^2(0,1), \mathbb{P})$. However, in [11, Section 7], Holmes argues that important generalization are needed for multivariate time series (and for functional time series even more so). An overview of this multivariate case can be found in [17] where the author stresses the importance of the Gramian structure of the product space $\mathcal{H}$. The Gramian matrix between two vectors $V = (V^{(1)}, \cdots, V^{(q)}) \in \mathcal{H}$ and $W = (W^{(1)}, \cdots, W^{(q)}) \in \mathcal{H}$ is the $q \times q$ matrix $[V, W]_{\mathcal{H}}$ with entries $\left(\left\langle V^{(k)}, W^{(l)}\right\rangle_{\mathcal{H}}\right)_{1\leq k,l\leq q}$ which coincides with the covariance matrix if

V or W are centered. Using this Gramian structure, Relations (1.1)–(1.4) are easily adapted by strengthening the weak stationarity to impose that $[X_s, X_t]_{\mathcal{H}}$ only depends on $s-t$ (see [17, Section 5]). This stronger weak stationarity assumption not only ensures that the lag operator U^X is (scalar product) isometric on H_X but also that it is Gramian-isometric on the larger space $\overline{\mathrm{Span}}\left(\mathrm{P}X_t\,,\, t \in \mathbb{Z}, \mathrm{P} \in \mathbb{C}^{q\times q}\right)$. Following the same approach, the development of a spectral theory of functional time series relies on exhibiting a Gramian structure for $\mathcal{H} = L^2(\Omega, \mathcal{F}, L^2(0,1), \mathbb{P})$ making it a *normal Hilbert module* and replacing the time domain space H_X by the *modular time domain*

$$\mathcal{H}^X = \overline{\mathrm{Span}}\left(\mathrm{P}X_t\,,\, t \in \mathbb{Z},\, \mathrm{P} \in \mathcal{L}_b(L^2(0,1))\right)\ , \tag{1.5}$$

where $\mathcal{L}_b(L^2(0,1))$ denotes the space of bounded operators on $L^2(0,1)$ onto itself. In comparison, in the definition of H_X used in [11], P is restricted to be a scalar operator. Thus, while H_X is a subspace of $\mathcal{H}$ seen as a Hilbert space, $\mathcal{H}^X$ is a submodule of $\mathcal{H}$ seen as a normal Hilbert module. Based on this simple fact, a natural path for achieving and fully exploiting a Cramér representation on $\mathcal{H}^X$ is:

Step 1) Interpret the representation (1.1) as the one of a Gramian-isometric operator on $\mathcal{H}^X$ (and not only an scalar product isometric operator on H_X).

Step 2) Deduce that the Cramér representation (1.2) can effectively be extended as a Gramian-isometric operator mapping $L^2(0,1) \to L^2(0,1)$-operator valued functions on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ to an element of $\mathcal{H}^X$.

Step 3) As a first consequence, the scalar product isometric relation (1.3) is extended to

$$[X_s, X_t]_{\mathcal{H}} = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda\,(s-t)}\ \nu(\mathrm{d}\lambda)\ , \quad s,t \in \mathbb{Z}\ , \tag{1.6}$$

where, here, ν is an operator valued measure on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$. This Gramian-isometric relationship corresponds to what is called the Herglotz theorem in the functional time series case.

Step 4) As a second consequence, the Cramér representation (1.4) of a linear filter is extended to the case where the transfer function Φ is now an $L^2(0,1) \to L^2(0,1)$-operator valued functions on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ (and not only a scalar valued functions on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$). This raises the question, in particular, of the precise condition required on the transfer function to replace the condition $\Phi \in L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu)$ of the scalar case.

Step 5) An interesting consequence of Step 4) is to study the composition of linear filters and deduce when and how it is possible to inverse them.

Step 6) An other interesting consequence of Step 2) is to derive the Cramér-Karhunen-Loève decomposition and the harmonic principal component analysis for any weakly stationary functional time series valued in a separable Hilbert space.

In this contribution, we basically follow this path, up to the following slight modifications.

1. We treat the more general case of a stochastic process $(X_t)_{t\in\mathrm{G}}$, where $(\mathrm{G}, +)$ is a locally compact Abelian (l.c.a.) group set of indices and for each $t \in \mathrm{G}$, X_t is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and valued in a separable Hilbert space $\mathcal{H}_0$ (endowed with its Borel σ-field). Typical examples for G and $\mathcal{H}_0$ are the ones of functional time series, namely $\mathrm{G} = \mathbb{Z}$ and $\mathcal{H}_0 = L^2(0,1)$ but, as far as spectral theory is concerned, the presentation of the results is not only more general (one can *e.g.* take $\mathrm{G} = \mathbb{R}$) but also more elegant in this general setting. We recall in Section 2.1 the definition of the dual group $\hat{\mathrm{G}}$ of continuous characters on G. Of course, in the discrete time case $\mathrm{G} = \mathbb{Z}$, any continuity condition imposed on a function defined on G is trivially satisfied. Such continuity conditions constitute a small price to pay (and the only one) in order to be able to treat the case of a general l.c.a. group G rather than focusing on the discrete time case alone.

2. For obvious practical reasons, it is usual to treat the mean of a stochastic process separately. Therefore we will assume that the process $(X_t)_{t\in\mathrm{G}}$ is centered.

3. We will consider the case where the separable Hilbert space $\mathcal{G}_0$ in which the output of the filter is valued is different from $\mathcal{H}_0$, the one of the input, that is, we replace $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0)$

in (1.5) by $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, the space of bounded operators from $\mathcal{H}_0$ to $\mathcal{G}_0$. This makes the results directly applicable in the case of different input and output spaces, especially in the case where they have different dimensions (so that they are not isomorphic).

The approach to derive a spectral theory following Step 1)– Step 4) is essentially contained in [13, 16, 12]. Our main contribution concerning these steps is to introduce all the preliminary definitions required to understand them, to select the most important results, to provide detailed proofs of the key points and to bring forward this approach in order to promote what we believe to be a more powerful, complete and easy to exploit approach than the more recently proposed ones in [19, 20, 26, 29, 30]. A very useful benefit of the Gramian-isometric approach is that it allows a concrete description of the spectral domain rather than relying on the completion of a pre-Hilbert space or on the compactification of a pointed convex cone as used in [26, Section 2.5] and [30], respectively. A greater benefit, however, is to make the Cramér representation much easier to exploit for deriving useful general results. This will be made apparent when establishing the composition and inversion of filters of Step 5), which to our best knowledge, appear to be novel in this degree of generality. Similarly, our versions of the Cramér-Karhunen-Loève decomposition and harmonic functional principal component analysis are not restricted to the case where the spectral density operator has none or finitely many points of discontinuity as in [26, 30].

As previously mentioned, each approach has its drawbacks and the main drawback of the one we are presenting here is probably that it requires lengthier, although not intrinsically difficult, preliminaries. In particular we need to precisely recall definitions of operator valued measures, operator valued functions (and the various notions of measurability related to them) and Gramian-isometric operators on normal Hilbert modules. All these basic definitions are assembled in Section 2 along with the useful facts about l.c.a. groups. Section 3 contains some preliminaries paving the way for describing the *modular spectral domain*. In particular, we explain how to use normal Hilbert modules for defining Gramian-orthogonally scattered measures. Section 4 contains the main results: 1) we offer a synthesis of the results of [13, 16, 12] providing a natural and complete spectral theory for weakly stationary processes valued in a separable Hilbert space; 2) then, this approach is exploited to address Step 5) and Step 6) above, successively; 3) in light of these results, we re-examine the differences with the approaches proposed in [26, 30]. All the postponed proofs are provided in Section 5 along with additional useful results.

Contents

2 Basic definitions and notation

2.1 Locally compact Abelian groups

A topological group is a group $(\mathrm{G},+)$ (with null element 0) endowed with a topology for which the addition and the inversion maps are continuous in $\mathrm{G}\times\mathrm{G}$ and G respectively. If G is Abelian (*i.e.* commutative) and is locally compact, Hausdorff for its topology, then it is called a locally compact Abelian (l.c.a.) group. A character χ of G is a group homomorphism from G to the unit circle group $\mathbb{U} := \{z \in \mathbb{C} : |z| = 1\}$ that is $\chi : \mathrm{G} \to \mathbb{U}$ and for all $s,t \in \mathrm{G}$, $\chi(s+t) = \chi(s)\chi(t)$. The dual group $\hat{\mathrm{G}}$ of an l.c.a. group G is the set of continuous characters of G. In particular, $\chi(0) = 1$ and $\overline{\chi(t)} = \chi(t)^{-1} = \chi(-t)$ for all $t \in \mathrm{G}$. $\hat{\mathrm{G}}$ is a multiplicative Abelian group if we define the product of $\chi_1, \chi_2 \in \hat{\mathrm{G}}$, as $\chi_1\chi_2 : t \mapsto \chi_1(t)\chi_2(t)$, the identity element as $\hat{e} : t \mapsto 1$ and the inverse of $\chi \in \hat{\mathrm{G}}$ as $\chi^{-1} : t \mapsto \chi(t)^{-1} = \overline{\chi(t)}$. $\hat{\mathrm{G}}$ becomes an l.c.a. group when endowed with the compact-open topology, that is the topology for which $\chi_n \to \chi$ in $\hat{\mathrm{G}}$ if and only if for every compact $K \subset \mathrm{G}$, $\chi_n \to \chi$ uniformly on K *i.e.* $\sup_{t\in K} |\chi_n(t) - \chi(t)| \xrightarrow[n\to+\infty]{} 0$.

A result known as the Pontryagin Duality Theorem (see [24, Theorem 1.7.2]) states that G and $\hat{\hat{\mathrm{G}}}$ are isomorphic via the evaluation map $\begin{array}{ccc} \mathrm{G} & \to & \hat{\hat{\mathrm{G}}} \\ t & \mapsto & \mathrm{e}_t \end{array}$ where $\mathrm{e}_t : \chi \mapsto \chi(t)$ in the sense that this map is a bijective continuous homomorphisms with continuous inverse. In particular, this means that $\{\mathrm{e}_t : t \in \mathrm{G}\}$ is the set of characters of $\hat{\mathrm{G}}$ (*i.e.* $\hat{\hat{\mathrm{G}}}$).

If $\mathrm{G} = \mathbb{Z}$ endowed with the addition of integers, the dual set $\hat{\mathbb{Z}}$ of characters contains all $\mathbb{Z} \to \mathbb{U}$-functions $\chi : t \mapsto z^t$ for some $z \in \mathbb{U}$. Since the compact sets of $\mathbb{Z}$ are the finite subsets of $\mathbb{Z}$, the compact-open topology on $\hat{\mathbb{Z}}$ is the same as the one induced by pointwise convergence. It is then easy to show that $\hat{\mathbb{Z}}$, $\mathbb{U}$ and $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ are isomorphic (from $\hat{\mathbb{Z}}$ to $\mathbb{U}$ take $\chi \mapsto \chi(1)$ and from $\mathbb{T}$ to $\mathbb{U}$ take $\lambda \mapsto \mathrm{e}^{\mathrm{i}\lambda}$). In this case we identify $\hat{\mathbb{Z}}$ with $\mathbb{T}$, often represented as $(-\pi,\pi]$ in the time series literature. This means that an integral on $\chi \in \hat{\mathbb{Z}}$ is replaced by an integral on $\lambda \in \mathbb{T}$ (or $\lambda \in (-\pi,\pi]$) with $\chi(t)$ replaced by $\mathrm{e}^{\mathrm{i}\lambda\, t}$ for all $t \in \mathbb{Z}$.

The other classical example of l.c.a. group is $\mathbb{R}$ endowed with usual addition and topology. Then the dual set $\hat{\mathbb{R}}$ contains all $\mathbb{R} \to \mathbb{U}$-functions $\chi : t \mapsto \mathrm{e}^{\mathrm{i}t\lambda}$ for some $\lambda \in \mathbb{R}$ (see for example [6, Theorem 9.11.]). Then $\hat{\mathbb{R}}$ and $\mathbb{R}$ are isomorphic via the mapping $\lambda \mapsto (t \mapsto \mathrm{e}^{\mathrm{i}t\lambda})$ and it is usual to identify $\hat{\mathbb{R}}$ with $\mathbb{R}$.

2.2 Operator spaces

In this section, we recall basic definitions on linear operators which can be found, for example, in [32]. Let $\mathcal{H}$ and $\mathcal{G}$ be two (complex) Hilbert spaces. The scalar product and norm, e.g. associated to $\mathcal{H}$, are denoted by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$. Let $\mathcal{O}(\mathcal{H},\mathcal{G})$ denote the set of linear operators P from $\mathcal{H}$ to $\mathcal{G}$ whose domains, denoted by $\mathcal{D}(\mathrm{P})$, are linear subspaces of $\mathcal{H}$. We then denote by $\mathcal{L}_b(\mathcal{H},\mathcal{G})$ its subset of continuous operators, by $\mathcal{K}(\mathcal{H},\mathcal{G})$ its subset of compact continuous operators and, for all $p \in [1,\infty)$, by $\mathcal{S}_p(\mathcal{H},\mathcal{G})$ the Schatten-p class of compact operators with ℓ^p singular values. If $\mathcal{G} = \mathcal{H}$, we omit $\mathcal{G}$ in the notation of these operator sets. We denote by $\|\cdot\|$ the operator norm on $\mathcal{L}_b(\mathcal{H},\mathcal{G})$ and by $\|\cdot\|_p$ the Schatten-p norm on $\mathcal{S}_p(\mathcal{H},\mathcal{G})$. For all $\mathrm{P} \in \mathcal{S}_1(\mathcal{H})$, we denote the trace of P by $\mathrm{Tr}(\mathrm{P})$. Schatten-1 and Schatten-2 operators are usually referred to as *trace-class* and *Hilbert-Schmidt* operators respectively. For any $\mathrm{P} \in \mathcal{L}_b(\mathcal{H},\mathcal{G})$ we denote its adjoint by P^{H} (which is compact if P is compact). An operator of $\mathcal{L}_b(\mathcal{H})$ is said to be auto-adjoint if it is equal to its adjoint. For all $x \in \mathcal{H}$ and $y \in \mathcal{G}$, we denote by $x \otimes y$ the trace-class operator from $\mathcal{G}$ onto $\mathcal{H}$ defined by $(x \otimes y)z = \langle z, y\rangle_{\mathcal{G}}\, x$ for all $z \in \mathcal{G}$. As usual, we identify an element $x \in \mathcal{H}$ with the mapping $z \mapsto zx$ from $\mathbb{C}$ onto $\mathcal{H}$, so that $x^{\mathsf{H}} : y \mapsto \langle y, x\rangle_{\mathcal{H}}$ is seen as an operator of $\mathcal{H}$ onto $\mathbb{C}$. In particular, we can write for $x \in \mathcal{H}$ and $y \in \mathcal{G}$, $x \otimes y = xy^{\mathsf{H}}$. An operator $\mathrm{P} \in \mathcal{L}_b(\mathcal{H})$, is said to be *positive* (denoted by $P \succeq 0$) if for all $x \in \mathcal{H}$, $\langle \mathrm{P}x, x\rangle_{\mathcal{H}} \geq 0$ and we denote by $\mathcal{L}_b^+(\mathcal{H})$, $\mathcal{K}^+(\mathcal{H})$ and $\mathcal{S}_p^+(\mathcal{H})$ the sets of positive, positive compact and positive Schatten-p operators. Any positive operator is auto-adjoint. If

$\mathrm{P} \in \mathcal{K}^+(\mathcal{H})$ then there exists a unique operator of $\mathcal{K}^+(\mathcal{H})$, denoted by $\mathrm{P}^{1/2}$, which satisfies $\mathrm{P} = \left(\mathrm{P}^{1/2}\right)^2$. We define the *absolute value* of P as the operator $|\mathrm{P}| := \left(\mathrm{P}^{\mathsf{H}}\mathrm{P}\right)^{1/2} \in \mathcal{K}^+(\mathcal{H})$.

Finally, we recall the definition of the *weak operator topology* (w.o.t.). A sequence $(\mathrm{P}_n)_{n\in\mathbb{N}} \in \mathcal{L}_b(\mathcal{H},\mathcal{G})^{\mathbb{N}}$ converges to an operator $\mathrm{P} \in \mathcal{L}_b(\mathcal{H},\mathcal{G})$ in w.o.t. if for all $x \in \mathcal{H}$ and $y \in \mathcal{G}$, $\lim_{n\to+\infty} \langle \mathrm{P}_n x, y\rangle_{\mathcal{G}} = \langle \mathrm{P}x, y\rangle_{\mathcal{G}}$. Using the polarization identity

$$\langle \mathrm{P}_n x, y\rangle_{\mathcal{G}} = \frac{1}{4}\left(\langle \mathrm{P}_n(x+y),(x+y)\rangle_{\mathcal{G}} - \langle \mathrm{P}_n(x-y),(x-y)\rangle_{\mathcal{G}} \right. \\ \left. + \mathrm{i}\,\langle \mathrm{P}_n(x+\mathrm{i}y),(x+\mathrm{i}y)\rangle_{\mathcal{G}} - \mathrm{i}\,\langle \mathrm{P}_n(x-\mathrm{i}y),(x-\mathrm{i}y)\rangle_{\mathcal{G}}\right) ,$$

to get the convergence in w.o.t., it is sufficient to show that $\lim_{n\to\infty} \langle \mathrm{P}_n x, x\rangle_{\mathcal{G}} = \langle \mathrm{P}x, x\rangle_{\mathcal{G}}$ for all $x \in \mathcal{H}$.

2.3 Integration of functions valued in a vector or operator space

Let $(\Lambda,\mathcal{A})$ be a measurable space and $(E, \|\cdot\|_E)$ be a Banach space. A function $f : \Lambda \mapsto E$ is said to be *measurable* if it is the pointwise limit of a sequence of E-valued *simple functions*, *i.e.* a sequence valued in $\mathrm{Span}\left(\mathbb{1}_A x \,:\, A \in \mathcal{A},\, x \in E\right)$. When E is separable, this notion is equivalent to the usual Borel-measurability, (*i.e.* to having $f^{-1}(A) \in \mathcal{A}$ for all $A \in \mathcal{B}(E)$, the Borel σ-field on E) and to the measurability of $\phi \circ f$ for all $\phi \in E^*$ (where E^* is the continuous dual of E). This last equivalence is known as Petti's measurability theorem and is proven in [21]. We denote by $\mathbb{F}(\Lambda,\mathcal{A},E)$ the space of measurable functions from Λ to E. For a non-negative measure μ on $(\Lambda,\mathcal{A})$ and $p \in [1,\infty]$, we denote by $\mathcal{L}^p(\Lambda,\mathcal{A},E,\mu)$ the space of functions $f \in \mathbb{F}(\Lambda,\mathcal{A},E)$ such that $\int \|f\|_E^p \,\mathrm{d}\mu$ (or μ-essup $\|f\|_E$ for $p = \infty$) is finite and by $L^p(\Lambda,\mathcal{A},E,\mu)$ its quotient space with respect to μ-a.e. equality. The corresponding norm is denoted by $\|\cdot\|_{L^p(\Lambda,\mathcal{A},E,\mu)}$. The Bochner integral is defined on $L^1(\Lambda,\mathcal{A},E,\mu)$ by linear and continuous extension of the mapping $\mathbb{1}_A x \to \mu(A)\,x$ defined for $x \in E$ and $A \in \mathcal{A}$ such that $\mu(A) < \infty$.

In the particular case where E is a space of linear operators between two Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$, we introduce the following weaker notion of measurability.

Definition 2.1 (Simple measurability)**.** *A function $\Phi : \Lambda \to \mathcal{L}_b(\mathcal{H},\mathcal{G})$ is said to be* simply measurable *if for all $x \in \mathcal{H}$, $\lambda \mapsto \Phi(\lambda)x$ is measurable as a $\mathcal{G}$-valued function. The set of such functions is denoted by $\mathbb{F}_s(\Lambda,\mathcal{A},\mathcal{H},\mathcal{G})$ or simply $\mathbb{F}_s(\Lambda,\mathcal{A},\mathcal{H})$ if $\mathcal{G} = \mathcal{H}$.*

Note that for all Banach spaces $\mathcal{E}$ which are continuously embedded in $\mathcal{L}_b(\mathcal{H},\mathcal{G})$ (*e.g.* $\mathcal{S}_p(\mathcal{H},\mathcal{G})$ for $p \geq 1$ or $\mathcal{K}(\mathcal{H},\mathcal{G})$), the following inclusions hold

$$\mathbb{F}(\Lambda,\mathcal{A},\mathcal{E}) \subset \mathbb{F}_s(\Lambda,\mathcal{A},\mathcal{H},\mathcal{G}) \ . \tag{2.1}$$

When $\mathcal{H}$ and $\mathcal{G}$ are separable, the converse inclusion holds for $\mathcal{E} = \mathcal{K}(\mathcal{H},\mathcal{G})$ and for $\mathcal{E} = \mathcal{S}_p(\mathcal{H},\mathcal{G})$ with $p \in \{1,2\}$, see Lemma 5.1.

The space $\mathcal{O}(\mathcal{H},\mathcal{G})$, is not a Banach space but we can still define measurability in the following sense, which we slightly adapted from [16], [12, Section 3.4].

Definition 2.2 ($\mathcal{O}$-measurability)**.** *A function $\Phi : \Lambda \to \mathcal{O}(\mathcal{H},\mathcal{G})$ is said to be $\mathcal{O}$-measurable if it satisfies the two following conditions.*

(i) For all $x \in \mathcal{H}$, $\{\lambda \in \Lambda \,:\, x \in \mathcal{D}(\Phi(\lambda))\} \in \mathcal{A}$.

(ii) There exist a sequence $(\Phi_n)_{n\in\mathbb{N}}$ valued in $\mathbb{F}_s(\Lambda,\mathcal{A},\mathcal{H},\mathcal{G})$ such that for all $\lambda \in \Lambda$ and $x \in \mathcal{D}(\Phi(\lambda))$, $\Phi_n(\lambda)x$ converges to $\Phi(\lambda)x$ in $\mathcal{G}$ as $n \to \infty$.

We denote by $\mathbb{F}_{\mathcal{O}}(\Lambda,\mathcal{A},\mathcal{H},\mathcal{G})$ the space of such functions Φ.

Clearly, we have $\mathbb{F}_s(\Lambda,\mathcal{A},\mathcal{H},\mathcal{G}) \subset \mathbb{F}_{\mathcal{O}}(\Lambda,\mathcal{A},\mathcal{H},\mathcal{G})$.

2.4 Vector valued and Positive Operator Valued Measures

A measure μ defined on the measurable space $(\Lambda,\mathcal{A})$ and valued in the Banach space $(E, \|\cdot\|_E)$ is an $\mathcal{A} \to E$-mapping such that, for any sequence $(A_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ of pairwise disjoint sets, $\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$, where the series converges in E, that is,

$$\lim_{N\to+\infty} \left\| \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) - \sum_{n=0}^{N} \mu(A_n) \right\|_E = 0 \, .$$

For such a measure μ, the mapping

$$\|\mu\|_E : A \mapsto \sup\left\{\sum_{i\in\mathbb{N}} \|\mu(A_i)\|_E \,:\, (A_i)_{i\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}} \text{ is a countable partition of } A\right\}$$

defines a non-negative measure on $(\Lambda, \mathcal{A})$ called the *variation measure* of μ. For instance, if $E = \mathcal{S}_1(\mathcal{H})$, we write $\|\mu\|_1$ since we use $\|\cdot\|_1$ to denote the Shatten-1 norm. Integrals of functions in $L^1(\Lambda, \mathcal{A}, \|\mu\|_E)$ with respect to μ are defined by first considering simple functions and by extending the obtained linear form to the whole space $L^1(\Lambda, \mathcal{A}, \|\mu\|_E)$ by continuity, see [9, P. 120]. When Λ is a locally-compact topological space and $\mathcal{A}$ is the Borel σ-field, an E-valued measure μ is said to be *regular* if for all $A \in \mathcal{A}$ and $\epsilon > 0$, there exist a compact set $K \in \mathcal{A}$ and an open set $U \in \mathcal{A}$ with $K \subset A \subset U$ such that $\|\mu(U \setminus K)\|_E \leq \epsilon$. The definition of regularity for non-finite (non-negative) measures is similar but the property is only required for A such that $\mu(A) < +\infty$. From the straightforward inequality $\|\mu(A)\|_E \leq \|\mu\|_E(A)$ for all $A \in \mathcal{A}$, we get that if μ is an E-valued measure with finite and regular variation, then μ is regular.

As for functions, the special case of operator-valued measures is of interest. In particular Positive Operator Valued Measures (p.o.v.m.'s) which are widely used in Quantum Mechanics. Here, we provide useful definitions and results for our purpose and refer to [3] for details.

Definition 2.3 (Positive Operator Valued Measures (p.o.v.m.))**.** *Let $(\Lambda, \mathcal{A})$ be a measurable space and $\mathcal{H}$ be a Hilbert space. A Positive Operator Valued Measure (p.o.v.m.) on $(\Lambda, \mathcal{A}, \mathcal{H})$ is a mapping $\nu : \mathcal{A} \to \mathcal{L}_b^+(\mathcal{H})$ such that for all sequences of disjoint sets $(A_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$,*

$$\nu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \nu(A_n) \tag{2.2}$$

where the series converges in $\mathcal{L}_b^+(\mathcal{H})$ in w.o.t.

It is interesting to note that, due to properties of positive operators, the convergence of the series in (2.2) in w.o.t. implies its convergence in s.o.t.(see [3, Proposition 1]). However, the series does not necessarily converge in operator norm which implies that, in this definition, a p.o.v.m. does not need to be an $\mathcal{L}_b(\mathcal{H})$-valued measure. Therefore the above definitions of integrals and regularity cannot be applied. This is circumvented by noting that a p.o.v.m. is entirely characterized by the family of non-negative measures $\left\{\nu_x : A \mapsto x^{\mathsf{H}}\nu(A)x \,:\, x \in \mathcal{H}\right\}$. Based on this characterization, we introduce two definitions related to p.o.v.m.'s, the first one about the regularity property and the second one about integrals of bounded scalar valued function.

Definition 2.4 (Regular p.o.v.m.)**.** *Let Λ be a locally-compact topological space with Borel σ-field $\mathcal{A}$ and $\mathcal{H}$ be a Hilbert space. Then a p.o.v.m. ν on $(\Lambda, \mathcal{A}, \mathcal{H})$ is said to be* regular *if for all $x \in \mathcal{H}$, the non-negative measure $\nu_x : A \mapsto x^{\mathsf{H}}\nu(A)x$ is regular.*

An alternative equivalent definition of regular p.o.v.m.'s is [3, Definition 14], see also Theorem 20 in the same reference. We now define the integral of bounded function with respect to a p.o.v.m.. This will be used in the general Bochner theorem (later stated as Theorem 4.6).

Definition 2.5 (Integral of a scalar valued function with respect to a p.o.v.m.)**.** *Let $(\Lambda, \mathcal{A})$ be a measurable space, $\mathcal{H}$ be a Hilbert space, ν be a p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H})$ and define for all $x \in \mathcal{H}$, the non-negative measure $\nu_x : A \mapsto x^{\mathsf{H}}\nu(A)x$. Let $f : \Lambda \to \mathbb{C}$ be a bounded and measurable function. Then the integral of f with respect to ν is the unique operator in $\mathcal{L}_b(\mathcal{H})$, denoted by $\int f(\lambda)\,\nu(\mathrm{d}\lambda)$, such that for all $x \in \mathcal{H}$,*

$$x^{\mathsf{H}}\left(\int f(\lambda)\,\nu(\mathrm{d}\lambda)\right)x = \int f(\lambda)\,\nu_x(\mathrm{d}\lambda)\,.$$

The existence of the autoadjoint integral operator $\int f(\lambda)\,\nu(\mathrm{d}\lambda)$ in Definition 2.5 through the mapping $x \mapsto x^{\mathsf{H}}\left(\int f(\lambda)\,\nu(\mathrm{d}\lambda)\right)x$ is straightforward, see [3, Theorem 9]. The integral in Definition 2.5 is only valid for bounded functions. Generalizing this integral to unbounded functions is complicated. Nevertheless, when dealing with spectral operator measures of weakly stationary processes valued in a separable Hilbert space, we can rely on the additional trace-class property, which makes all the previous definitions easier to handle and extend. Hereafter, if $\mathcal{H}_0$ is a separable Hilbert space, we provide the definition of a trace-class p.o.v.m., derive its interpretation as an $\mathcal{S}_1(\mathcal{H}_0)$-valued measure, state the Radon-Nikodym property that they enjoy and extend the definition of the integral of a scalar valued function in this context.

Definition 2.6 (Trace-class p.o.v.m.)**.** *Let $(\Lambda, \mathcal{A})$ be a measurable space, $\mathcal{H}_0$ be a* separable *Hilbert space and ν be a p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. We say that ν is a* trace-class*-p.o.v.m. if it is $\mathcal{S}_1^+(\mathcal{H}_0)$-valued.*

The first advantage of a trace-class p.o.v.m. is that it fits the framework of vector-valued measures, namely, we have the following result, whose proof can be found in Section 5.1.

Lemma 2.1. *Let $(\Lambda, \mathcal{A})$ be a measurable space and $\mathcal{H}_0$ be a* separable *Hilbert space. Then a p.o.v.m. ν on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ is trace-class if and only if $\nu(\Lambda) \in \mathcal{S}_1(\mathcal{H}_0)$. In this case, ν is an $\mathcal{S}_1(\mathcal{H}_0)$-valued measure (in the sense that* (2.2) *holds in $\|\cdot\|_1$-norm) with finite variation measure $\|\nu\|_1 : A \mapsto \|\nu(A)\|_1$. Moreover, regularity of ν as a p.o.v.m. is equivalent to regularity of ν as an $\mathcal{S}_1(\mathcal{H}_0)$-valued measure which is itself equivalent to regularity of $\|\nu\|_1$.*

Another advantage of trace-class p.o.v.m.'s is that they satisfy the following Radon-Nikodym property, whose proof can be found in Section 5.1.

Theorem 2.2. *Let $(\Lambda, \mathcal{A})$ be a measure space, $\mathcal{H}_0$ a* separable *Hilbert space and ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. Let μ be a σ-finite non-negative measure on $(\Lambda, \mathcal{A})$. Then $\|\nu\|_1 \ll \mu$ (i.e. for all $A \in \mathcal{A}$, $\mu(A) = 0 \Rightarrow \|\nu\|_1(A) = 0$), if and only if there exists $g \in L^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{H}_0), \mu)$ such that $\mathrm{d}\nu = g \, \mathrm{d}\mu$, i.e. for all $A \in \mathcal{A}$,*

$$\nu(A) = \int_A g \, \mathrm{d}\mu \,. \tag{2.3}$$

In this case, g is unique and is called the density of ν with respect to μ and we write

$$g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \,.$$

Moreover, the following assertions hold.

(a) For μ-almost every $\lambda \in \Lambda$, $g(\lambda) \in \mathcal{S}_1^+(\mathcal{H}_0)$.

(b) The mapping $g^{1/2} : \lambda \mapsto g(\lambda)^{1/2}$ belongs to $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0), \mu)$.

(c) The density of $\|\nu\|_1$ with respect to μ is $\|g\|_1$. In particular, $g = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1} \|g\|_1$ μ-a.e. and if $\mu = \|\nu\|_1$, then $\|g\|_1 = 1$ μ-a.e.

(d) Let $f : \Lambda \to \mathbb{C}$ be measurable. Then $f \in L^1(\Lambda, \mathcal{A}, \|\nu\|_1)$ if and only if $\lambda \mapsto f(\lambda) \, g(\lambda) \in L^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{H}_0), \mu)$, and we have $\int f(\lambda) \, \nu(\mathrm{d}\lambda) = \int f(\lambda) \, g(\lambda) \, \mu(\mathrm{d}\lambda)$.

In Assertion (d), the first integral is that of a scalar-valued function with-respect to the $\mathcal{S}_1(\mathcal{H}_0)$-valued measure ν as recalled above for general vector-valued measures with finite variation and the second is the (Bochner) integral of an $\mathcal{S}_1(\mathcal{H}_0)$-valued function with-respect to the non-negative measure μ as recalled in Section 2.3. Of course, if f is bounded on Λ, these integrals coincide with the integral of f with respect to ν of Definition 2.5 in which ν is seen as a p.o.v.m. The Radon-Nikodym property of trace-class p.o.v.m.'s is a key step to extend such integrals to operator valued functions, hence allowing us to use a handy definition of the integral of an operator valued function with respect to an operator valued measure, in the particular case where this measure is a trace-class p.o.v.m. This will be done in Definition 3.3.

2.5 Normal Hilbert modules

Modules extend the notion of vector spaces to the case where scalar multiplication is replaced by a multiplicative operation with elements of a ring. The case where the ring is $\mathcal{L}_b(\mathcal{H}_0)$ for a separable Hilbert space $\mathcal{H}_0$ is of particular interest for $\mathcal{H}_0$-valued random variables. In short, a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module is a Hilbert space endowed with a *module action* and a *Gramian*. A Gramian $[\cdot, \cdot]$ is similar to a scalar product but is valued in the space $\mathcal{S}_1(\mathcal{H}_0)$ and is related to scalar product by the relation $\langle \cdot, \cdot \rangle = \mathrm{Tr}([\cdot, \cdot])$. Notions such as sub-modules, Gramian-orthogonality, Gramian-isometric operators are natural extensions of their counterparts in the Hilbert framework. We give such useful definitions hereafter and refer to [12, Chapter 2] for details.

Definition 2.7 ($\mathcal{L}_b(\mathcal{H}_0)$-module)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space. An $\mathcal{L}_b(\mathcal{H}_0)$-module is a commutative group $(\mathcal{H},+)$ such that there exists a multiplicative operation (called the* module action*)*

$$\begin{array}{lll} \mathcal{L}_b(\mathcal{H}_0)\times\mathcal{H} & \to & \mathcal{H} \\ (\mathrm{P},x) & \mapsto & \mathrm{P}\bullet x \end{array}$$

which satisfies the usual distributive properties : for all $\mathrm{P},\mathrm{Q}\in\mathcal{L}_b(\mathcal{H}_0)$, and $x,y\in\mathcal{H}$,

$$\begin{aligned} \mathrm{P}\bullet(x+y) &= \mathrm{P}\bullet x+\mathrm{P}\bullet y, \\ (\mathrm{P}+\mathrm{Q})\bullet x &= \mathrm{P}\bullet x+\mathrm{Q}\bullet x, \\ (\mathrm{PQ})\bullet x &= \mathrm{P}\bullet(\mathrm{Q}\bullet x), \\ \mathrm{Id}_{\mathcal{H}_0}\bullet x &= x. \end{aligned}$$

Next, we endow an $\mathcal{L}_b(\mathcal{H}_0)$-module with an $\mathcal{L}_b(\mathcal{H}_0)$-valued product.

Definition 2.8 ((Normal) pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space. We say that $(\mathcal{H},[\cdot,\cdot]_{\mathcal{H}})$ is a pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module if $\mathcal{H}$ is an $\mathcal{L}_b(\mathcal{H}_0)$-module and $[\cdot,\cdot]_{\mathcal{H}}:\mathcal{H}\times\mathcal{H}\to\mathcal{L}_b(\mathcal{H}_0)$ satisfies, for all $x,y,z\in\mathcal{H}$, and $\mathrm{P}\in\mathcal{L}_b(\mathcal{H}_0)$,*

(i) $[x,x]_{\mathcal{H}}\in\mathcal{L}_b^+(\mathcal{H}_0)$,

(ii) $[x,x]_{\mathcal{H}}=0$ *if and only if* $x=0$,

(iii) $[x+\mathrm{P}\bullet y,z]_{\mathcal{H}}=[x,z]_{\mathcal{H}}+\mathrm{P}[y,z]_{\mathcal{H}}$,

(iv) $[y,x]_{\mathcal{H}}=[x,y]_{\mathcal{H}}^{\mathsf{H}}$.

If moreover, for all $x,y\in\mathcal{H}$, $[x,y]_{\mathcal{H}}\in\mathcal{S}_1(\mathcal{H}_0)$, we say that $[\cdot,\cdot]_{\mathcal{H}}$ is a Gramian *and that $\mathcal{H}$ is a* normal *pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module.*

Note that an $\mathcal{L}_b(\mathcal{H}_0)$-module is a vector space if we define the scalar-vector multiplication by $\alpha x=(\alpha\mathrm{Id}_{\mathcal{H}_0})\bullet x$ for all $\alpha\in\mathbb{C}$, $x\in\mathcal{H}$ and that, in the particular case where $[\cdot,\cdot]_{\mathcal{H}}$ is a Gramian, then $\langle\cdot,\cdot\rangle_{\mathcal{H}}:=\mathrm{Tr}[\cdot,\cdot]_{\mathcal{H}}$ is a scalar product. Hence a normal pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module is also a pre-Hilbert space. We can now define the following.

Definition 2.9 (normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module)**.** *A normal pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module is said to be a* normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module *if it is complete (for the norm defined by $\|x\|_{\mathcal{H}}^2=\langle x,x\rangle_{\mathcal{H}}=\left\|[x,x]_{\mathcal{H}}\right\|_1$).*

Definition 2.10 (Submodules and $\mathcal{L}_b(\mathcal{H}_0)$-linear operators)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space and $\mathcal{H},\mathcal{G}$ be two $\mathcal{L}_b(\mathcal{H}_0)$-modules. Then a subset of $\mathcal{H}$ is called a* submodule *if it is an $\mathcal{L}_b(\mathcal{H}_0)$-module. An operator $\mathrm{F}\in\mathcal{L}_b(\mathcal{H},\mathcal{G})$ is said to be* $\mathcal{L}_b(\mathcal{H}_0)$-linear *if for all $\mathrm{P}\in\mathcal{L}_b(\mathcal{H}_0)$ and $x\in\mathcal{H}$, $\mathrm{F}(\mathrm{P}\bullet x)=\mathrm{P}\bullet(\mathrm{F}x)$. In the case where $\mathcal{H}$ is a normal pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module, we denote, for any $E\subset\mathcal{H}$, $\overline{\mathrm{Span}}^{\mathcal{H}}(E)$ the smallest linear subspace of $\mathcal{H}$ which contains E and is closed for the norm $\|\cdot\|_{\mathcal{H}}$. It is a submodule of $\mathcal{H}$.*

Definition 2.11 (Gramian-isometric operators)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space, $\mathcal{H},\mathcal{G}$ be two pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-modules and $U:\mathcal{H}\to\mathcal{G}$ an $\mathcal{L}_b(\mathcal{H}_0)$-linear operator. Then U is said to be*

(i) Gramian-isometric *if for all $x,y\in\mathcal{H}$, $[Ux,Uy]_{\mathcal{G}}=[x,y]_{\mathcal{H}}$,*

(ii) Gramian-unitary *if it is bijective Gramian-isometric.*

The space $\mathcal{H}$ is said to be Gramian-isometrically embedded *in $\mathcal{G}$ (denoted by $\mathcal{H}\underset{\approx}{\subseteq}\mathcal{G}$) if there exists a Gramian-isometric operator from $\mathcal{H}$ to $\mathcal{G}$. The spaces $\mathcal{H}$ and $\mathcal{G}$ are said to be* Gramian-isometrically isomorphic *(denoted by $\mathcal{H}\cong\mathcal{G}$) if there exists a Gramian-unitary operator from $\mathcal{H}$ to $\mathcal{G}$.*

The well-known isometric extension theorem can be straightforwardly generalized to the case of Gramian-isometric operators as stated in the following proposition.

Proposition 2.3 (Gramian-isometric extension)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space, $\mathcal{H}$ be a normal pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module, and $\mathcal{G}$ be a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module. Let $(v_j)_{j\in J}$ and $(w_j)_{j\in J}$ be two collections of vectors in $\mathcal{H}$ and $\mathcal{G}$ respectively with J an arbitrary index set. If for all $i,j\in J$, $[v_i,v_j]_{\mathcal{H}}=[w_i,w_j]_{\mathcal{G}}$ then there exists a unique Gramian-isometric operator*

$$S:\overline{\mathrm{Span}}^{\mathcal{H}}(\mathrm{P}\bullet v_j,\mathrm{P}\in\mathcal{L}_b(\mathcal{H}_0),j\in J)\to\mathcal{G}$$

such that for all $j \in J$, $Sv_j = w_j$. If moreover $\mathcal{H}$ is complete then

$$S\left(\overline{\operatorname{Span}}^{\mathcal{H}}\left(\mathrm{P} \bullet v_j, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0), j \in J\right)\right) = \overline{\operatorname{Span}}^{\mathcal{G}}\left(\mathrm{P} \bullet w_j, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0), j \in J\right)$$

Let us introduce three examples of normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-modules that will be of interest in the following.

Example 2.1 (Normal Hilbert module $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$ for a non-negative measure μ)**.** *Let μ be a non-negative measure on $(\Lambda, \mathcal{A})$ and $\mathcal{H}_0, \mathcal{G}_0$ be two separable Hilbert spaces. Then $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$ is an $\mathcal{L}_b(\mathcal{G}_0)$-module with module action defined for all $\mathrm{P} \in \mathcal{L}_b(\mathcal{G}_0)$ and $\Phi \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$ by $\mathrm{P} \bullet \Phi : \lambda \mapsto \mathrm{P}\Phi(\lambda)$. For all $\Phi, \Psi \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$, we have $\Phi\Psi^{\mathsf{H}} \in L^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{H}_0), \mu)$, and thus*

$$[\Phi, \Psi]_{L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)} := \int \Phi\Psi^{\mathsf{H}} \, \mathrm{d}\mu$$

is well defined in $\mathcal{S}_1(\mathcal{H}_0)$. It is easy to show that it is Gramian and that $\left(L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu), [\cdot, \cdot]_{L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)}\right)$ is a $\mathcal{L}_b(\mathcal{G}_0)$-module normal Hilbert module.

Taking $\mathcal{G}_0 = \mathbb{C}$ in the previous example amounts to replace $\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$ by $\mathcal{H}_0$, which, in the specific case where $(\Lambda, \mathcal{A}, \mu)$ is a probability space leads to the following.

Example 2.2 (Normal Hilbert module $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{H}_0$ be a separable Hilbert space. The Bochner space $L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ is the space of $\mathcal{H}_0$-valued random variables Y such that $\mathbb{E}\left[\|Y\|^2_{\mathcal{H}_0}\right] < +\infty$. Then the expectation of Y is the unique vector $\mathbb{E}[Y] \in \mathcal{H}_0$ satisfying*

$$\langle \mathbb{E}[Y], x\rangle_{\mathcal{H}_0} = \mathbb{E}\left[\langle Y, x\rangle_{\mathcal{H}_0}\right], \quad \textit{for all } x \in \mathcal{H}_0 ,$$

and the covariance operator between $Y, Z \in L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ is the unique linear operator $\operatorname{Cov}(Y, Z) \in \mathcal{L}_b(\mathcal{H}_0)$, satisfying

$$\langle \operatorname{Cov}(Y, Z)\, y, x\rangle_{\mathcal{H}_0} = \operatorname{Cov}\left(\langle Y, x\rangle_{\mathcal{H}_0}, \langle Z, y\rangle_{\mathcal{H}_0}\right), \quad \textit{for all } x, y \in \mathcal{H}_0 .$$

The space $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ of all centered random variables in $L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ is a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module for the module action defined for all $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0)$ and $X \in \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ by $\mathrm{P} \bullet X = \mathrm{P}X$, and the Gramian

$$[X, Y]_{\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})} = \operatorname{Cov}(X, Y) .$$

Our last example is a more general formulation of the space of transfer functions used in [26, Section 2.5] and [29, Appendix B.2.3] for filtering functional time series and is a natural extension of [17] where the case of (finite dimensional) multivariate time series is considered (see the definition of $\underline{L}_{2,\underline{M}}$ in this reference).

Example 2.3 (Normal pre-Hilbert module $\left(L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1), [\cdot, \cdot]_\nu\right)$ for a trace-class p.o.v.m. ν.)**.** *Let $(\Lambda, \mathcal{A})$ be a measurable space, $\mathcal{H}_0, \mathcal{G}_0$ be two separable Hilbert spaces and ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ with density f with respect to its finite variation $\|\nu\|_1$. Then $L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ is a normal pre-Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module with module action defined for all $\mathrm{P} \in \mathcal{L}_b(\mathcal{G}_0)$ and $\Phi \in L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ by $\mathrm{P} \bullet \Phi : \lambda \mapsto \mathrm{P}\Phi(\lambda)$, and Gramian defined for all $\Phi, \Psi \in L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ by*

$$[\Phi, \Psi]_\nu := \int \Phi\, f\, \Psi^{\mathsf{H}} \, \mathrm{d}\|\nu\|_1 . \tag{2.4}$$

Note that the $\mathcal{S}_1(\mathcal{H}_0)$-valued Bochner integral in the right-hand side of (2.4) is well defined because by Theorem 2.2 (c), we have $\|f\|_1 = 1$, $\|\nu\|_1$-a.e. and thus $\left\|\Phi f \Psi^{\mathsf{H}}\right\|_1 \leq \|\Phi\|_{\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)} \|\Psi\|_{\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)}$, $\|\nu\|_1$-a.e., which implies $\Phi\, f\, \Psi^{\mathsf{H}} \in L^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{H}_0), \|\nu\|_1)$.

For all $\Phi, \Psi \in L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$, we denote by $\langle \Phi, \Psi \rangle_\nu$ the scalar product associated to the Gramian $[\Phi, \Psi]_\nu$. It is different from the scalar product $\langle \Phi, \Psi \rangle_{L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)}$ endowing the same space. In particular we have that, for all $\Phi \in L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$,

$$\|\Phi\|_\nu^2 = \mathrm{Tr} \int \Phi\, f\, \Phi^{\mathsf{H}}\, \mathrm{d}\|\nu\|_1 = \int \mathrm{Tr} \left(\Phi\, f\, \Phi^{\mathsf{H}} \right) \, \mathrm{d}\|\nu\|_1 \leq \int \|\Phi\|^2_{\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)} \, \mathrm{d}\|\nu\|_1 = \|\Phi\|^2_{L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)} , \quad (2.5)$$

where we used again that $\|f\|_1 = 1$, $\|\nu\|_1$-a.e. It is easy to find Φ's for which the inequality is strict.

Example 2.3 is pivotal for defining the modular spectral domain of a weakly stationary process with spectral operator measure ν. However, it does not suffice to describe the whole spectral domain because, as already noted in [26, Section 2.5] in a similar case, this space, in general, is not complete. As a result, unfortunately, the spectral domain is more complicated for functional time series than for (finite dimensional) multivariate time series. Of course, as proposed in [26, Section 2.5], it is always possible to use topological completion under a well chosen norm. These ideas are in fact very similar to the ones of [13, 16, 12] with the exception that the latter references provide a more general framework and lead to a modular spectral domain which is an explicit set of operator-valued functions. We will follow this approach in Section 3.2.

3 Preliminaries

3.1 Countably additive Gramian-orthogonally scattered (c.a.g.o.s.) measures

In this section, we introduce the notion of random c.a.g.o.s. measures which will have an important role in the construction provided by [13, 16, 12]. The terminologies c.a.o.s. and c.a.g.o.s. are borrowed from Definition 3 in [12, Section 3.1]

Definition 3.1 ((Random) c.a.o.s. measures)**.** *Let $\mathcal{H}$ be a Hilbert space and $(\Lambda, \mathcal{A})$ be a measurable space. We say that $W : \mathcal{A} \to \mathcal{H}$ is a* countably additive orthogonally scattered *(c.a.o.s.) measure on $(\Lambda, \mathcal{A}, \mathcal{H})$ if it is an $\mathcal{H}$-valued measure on $(\Lambda, \mathcal{A})$ such that for all $A, B \in \mathcal{A}$,*

$$A \cap B = \emptyset \Rightarrow \langle W(A), W(B) \rangle_{\mathcal{H}} = 0 \, .$$

In this case, the mapping

$$\nu_W : A \mapsto \langle W(A), W(A) \rangle_{\mathcal{H}}$$

is a finite non-negative measure on $(\Lambda, \mathcal{A})$ called the intensity measure *of W and we have that, for all $A, B \in \mathcal{A}$,*

$$\nu_W(A \cap B) = \langle W(A), W(B) \rangle_{\mathcal{H}} \, . \quad (3.1)$$

We say that W is regular if ν_W is regular. When $\mathcal{H}$ is the space $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ of Example 2.2, we say that W is an $\mathcal{H}_0$-valued random c.a.o.s. measure *on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$.*

The generalization to a normal Hilbert module is straightforward.

Definition 3.2 ((Random) c.a.g.o.s. measures)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space, $\mathcal{H}$ be a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module and $(\Lambda, \mathcal{A})$ be a measurable space. We say that $W : \mathcal{A} \to \mathcal{H}$ is a* countably additive Gramian-orthogonally scattered *(c.a.g.o.s.) measure on $(\Lambda, \mathcal{A}, \mathcal{H})$ if it is an $\mathcal{H}$-valued measure on $(\Lambda, \mathcal{A})$ such that for all $A, B \in \mathcal{A}$,*

$$A \cap B = \emptyset \Rightarrow [W(A), W(B)]_{\mathcal{H}} = 0 \, .$$

In this case, the mapping

$$\nu_W : A \mapsto [W(A), W(A)]_{\mathcal{H}}$$

is a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ called the intensity operator measure *of W and we have that, for all $A, B \in \mathcal{A}$,*

$$\nu_W(A \cap B) = [W(A), W(B)]_{\mathcal{H}} \, . \quad (3.2)$$

We say that W is regular if $\|\nu_W\|_1$ is regular. When $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ of Example 2.2, we say that W is an $\mathcal{H}_0$-valued random c.a.g.o.s. measure *on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$.*

The following remark will be useful.

Remark 3.1. *Recall that any $\mathcal{H}$-valued measure W is σ-additive in the sense that for any finite or countable collection $(A_j)_{j\in J} \in \Lambda^J$ of pairwise disjoint sets we have*

$$W\left(\bigcup_{j\in J} A_j\right) = \sum_{j\in J} W(A_j)\ ,$$

where, in the case where J is countably infinite, the infinite sum converges in $\mathcal{H}$ inconditionally. When W is a c.a.o.s., the summands are moreover orthogonal. When it is a c.a.g.o.s., they are Gramian-orthogonal.

It is easy to show that a c.a.o.s. measure W as in Definition 3.1 can be equivalently seen as the restriction of an isometric operator I from $L^2(\Lambda, \mathcal{A}, \nu_W)$ onto $\mathcal{H}$ by setting

$$W(A) = I(\mathbb{1}_A)\ , \qquad A \in \Lambda\ .$$

This simply follows by interpreting the left-hand side of (3.1) as the scalar product between $\mathbb{1}_A$ and $\mathbb{1}_B$ in $L^2(\Lambda, \mathcal{A}, \nu_W)$ so that I above can be defined as the unique isometric extension from $L^2(\Lambda, \mathcal{A}, \nu_W)$ to $\mathcal{H}$ of the isometric mapping defined by $\mathbb{1}_A \mapsto W(A)$ for $A \in \Lambda$. This observation gives a rigorous meaning to the integral in the Cramér representation (1.2) where $\hat{X}$ is c.a.o.s. (see [11, Section 2]). Similarly, if W is a c.a.g.o.s. measure as in Definition 3.2, the mapping defined by $\mathbb{1}_A\mathrm{P} \mapsto \mathrm{P}W(A)$ for $A \in \Lambda$ and $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0)$ is Gramian-isometric from the normal pre-Hilbert module $L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0), \|\nu_W\|_1)$ defined in Example 2.3 onto $\mathcal{H}$. Using Proposition 2.3, we get a Gramian-isometric extension on the whole space. In the case where $\mathcal{H}_0$ has finite dimension, this observation is a key step to derive a Cramér representation of the type (1.2) where $(X_t)_{t\in\mathbb{Z}}$ is a multivariate time series and $\hat{X}$ is c.a.g.o.s. (see [17]). In the infinite dimensional case, this Gramian-isometric extension is more useful if, before that, we complete the normal pre-Hilbert module $L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0), \|\nu_W\|_1)$, that is, we exhibit the smallest normal Hilbert module containing it. To do this, we will rely on the space $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ defined for a trace-class p.o.v.m. ν in the following section. Before that, let us note that in the case of random c.a.g.o.s. measure W, by definition of $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ in Example 2.2, the identity (3.2) shows that the covariance structure of the centered process $(W(A))_{A\in\mathcal{A}}$ is entirely determined by ν_W. The Gaussian case is interesting as it provides a way to build W from its intensity measure. In particular, the following result will be useful.

Theorem 3.1. *Let $\mathcal{H}_0$ be a separable Hilbert space and $(\Lambda, \mathcal{A})$ be a measurable space. Let ν be a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. Then there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an $\mathcal{H}_0$-valued random c.a.g.o.s. W on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure ν such that the process $(\langle W(A), x\rangle)_{A\in\mathcal{A}, x\in\mathcal{H}_0}$ is a (complex) Gaussian process.*

Proof. Define $\gamma : (\mathcal{H}_0 \times \mathcal{A})^2 \to \mathbb{C}$ by of

$$\gamma((x, A); (y, B)) = x^{\mathsf{H}}\nu(A \cap B)y = \left[x^{\mathsf{H}}\mathbb{1}_A, y^{\mathsf{H}}\mathbb{1}_B\right]_{\nu}\ ,$$

where we used the Gramian (2.4) of Example 2.3 with $\mathcal{G}_0 = \mathbb{C}$. Then it is easy to see γ is hermitian non-negative definite in the sense that for all $n \geq 1$, $x_1, \dots, x_n \in \mathcal{H}_0$, $A_1, \dots, A_n \in \mathcal{A}$ and $a_1, \dots, a_n \in \mathbb{C}$,

$$\sum_{i,j=1}^{n} a_i\overline{a_j}\gamma((x_i, A_i); (a_j, A_j)) \geq 0\ .$$

Let $(Z_{x,A})_{(x,A)\in\mathcal{H}_0\times\mathcal{A}}$ be the centered circularly-symmetric Gaussian process complex with covariance γ. Let $(\phi_n)_{0\leq n<N}$ be a Hilbert basis of $\mathcal{H}_0$, with $N = \dim \mathcal{H}_0 \in \{1, 2, \dots, \infty\}$. It is straightforward to show that for all $A \in \mathcal{A}$,

$$W(A) := \sum_{0\leq n<N} Z_{\phi_n, A}\, \phi_n$$

is well defined in $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ and that the so defined W is a random c.a.g.o.s. with intensity operator measure ν. ◻

3.2 The space $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$

As discussed in the previous sections, the role of c.a.o.s. and c.a.g.o.s. measures in the spectral theory of weakly stationary processes relies on their characterization by unitary or Gramian-unitary operators between the (modular) time domain and the (modular) spectral domain. This has been entirely studied in the case of univariate and multivariate time series, see [11] and [17], respectively, and the references therein. For time series valued in a general separable Hilbert space, defining the modular spectral domain requires to exhibit a suitable completion of the normal pre-Hilbert module $L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ of Example 2.3 where ν is a trace-class p.o.v.m. . In this section, we define such a space of operator-valued functions which are *square-integrable* with respect to p.o.v.m. ν. It was introduced in [16] and includes the space $L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ but is in general larger in the case where $\mathcal{H}_0$ has infinite dimension.

Definition 3.3. *Let $(\Lambda, \mathcal{A})$ be a measurable space, $\mathcal{H}_0, \mathcal{G}_0, \mathcal{I}_0$ be three separable Hilbert spaces and ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ with density f with respect to its finite variation $\|\nu\|_1$, as defined in Theorem 2.2. Then, we say that $(\Phi, \Psi) \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0) \times \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{I}_0)$ is* ν-integrable *if the three following assertions hold.*

(i) We have $\mathrm{Im}(f^{1/2}) \subset \mathcal{D}(\Phi)$ and $\mathrm{Im}(f^{1/2}) \subset \mathcal{D}(\Psi)$, $\|\nu\|_1$-a.e.

(ii) We have $\Phi f^{1/2} \in \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$ and $\Psi f^{1/2} \in \mathcal{S}_2(\mathcal{H}_0, \mathcal{I}_0)$, $\|\nu\|_1$-a.e.

(iii) We have $(\Phi f^{1/2})(\Psi f^{1/2})^{\mathsf{H}} \in \mathcal{L}^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{I}_0, \mathcal{G}_0), \|\nu\|_1)$.

In the case, we define

$$\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} := \int (\Phi f^{1/2})(\Psi f^{1/2})^{\mathsf{H}} \, \mathrm{d}\|\nu\|_1 \in \mathcal{S}_1(\mathcal{I}_0, \mathcal{G}_0) \; . \tag{3.3}$$

Moreover, we say that $\Phi \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$ is square ν-integrable *if (Φ, Φ) is ν-integrable and we denote by $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ the space of square ν-integrable functions in $\mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$.*

Remark 3.2. *Let us briefly comment this definition.*

1) *In (3.3), using the Radon-Nikodym property of the trace-class p.o.v.m. ν, we have thus defined an integral of operator-valued functions with respect to an operator valued measure as a simple Bochner theorem in $\mathcal{S}_1(\mathcal{I}_0, \mathcal{G}_0)$. By Theorem 2.2 (d), for a measurable scalar function $\phi : \Lambda \to \mathbb{C}$ we can interpret the integral $\int \phi \mathrm{d}\nu$ in which ν is seen as $\mathcal{S}_1(\mathcal{H}_0)$-valued measure as in Section 2.4 as the same integral as in (3.3) with $\Phi : \lambda \mapsto \phi(\lambda)\mathrm{Id}_{\mathcal{H}_0}$ and $\Psi \equiv \mathrm{Id}_{\mathcal{H}_0}$. Hence the integral (3.3) of Definition 3.3 can be seen as an extension of the integral of scalar-valued functions to operator-valued functions, with respect to a trace-class p.o.v.m.*

2) *It is easy to show that for all $\Phi, \Psi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$, (Φ, Ψ) is ν-integrable and thus $\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}}$ is well defined as above.*

3) *In the special case where Φ and Ψ are valued in $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, $\mathcal{O}$-measurability reduces to simple-measurability, (i) and (ii) are always verified, (iii) is equivalent to $\Phi f \Psi^{\mathsf{H}} \in \mathcal{L}^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{G}_0), \|\nu\|_1)$ and we get*

$$\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} = \int \Phi f \Psi^{\mathsf{H}} \, \mathrm{d}\|\nu\|_1 \; .$$

4) *Recall that, as explained in Example 2.3, for all $\Phi, \Psi \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$, we have $\Phi f \Psi^{\mathsf{H}} \in \mathcal{L}^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{G}_0), \|\nu\|_1)$. We thus get that*

$$\mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1) \subset \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) \; ,$$

and the Gramian $[\Phi, \Psi]_\nu$ defined on the smaller space as in (2.4) coincides with $\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}}$ defined in (3.3).

The following theorem, whose proof can be found in Section 5.2, shows that the same Gramian can be used over the larger space $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ and that it makes this space a normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module when quotiented by the set with zero norm.

Theorem 3.2. *Let $\mathcal{H}_0, \mathcal{G}_0$ be separable Hilbert spaces, $(\Lambda, \mathcal{A})$ a measurable space, ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ and $f = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$. Then $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ is an $\mathcal{L}_b(\mathcal{G}_0)$-module with module action*

$$\mathrm{P} \bullet \Phi : \lambda \mapsto \mathrm{P}\Phi(\lambda), \quad \mathrm{P} \in \mathcal{L}_b(\mathcal{G}_0), \Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) .$$

Moreover, we can endow $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ with the pseudo-Gramian

$$[\Phi, \Psi]_\nu := \int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} \quad \Phi, \Psi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) . \tag{3.4}$$

Then, for all $\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$, we have

$$\|\Phi\|_\nu = \left\|[\Phi, \Phi]_\nu\right\|_1^{1/2} = 0 \Longleftrightarrow \Phi f^{1/2} = 0 \quad \|\nu\|_1\text{-a.e.}$$

Let us denote the class of such Φ's by $\{\|\cdot\|_\nu = 0\}$ and the quotient space by

$$\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) := \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) \Big/ \{\|\cdot\|_\nu = 0\} .$$

Then $\left(\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu), [\cdot, \cdot]_\nu\right)$ is a normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module.

Clearly, the normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module $\left(\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu), [\cdot, \cdot]_\nu\right)$ contains the pre-Hilbert one of Example 2.3. The next result, whose proof can be found in Section 5.2, says that it is the smallest one.

Theorem 3.3. *Let $\mathcal{H}_0, \mathcal{G}_0$ be two separable Hilbert spaces, $(\Lambda, \mathcal{A})$ a measurable space, and ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. Then the space $L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ is dense in $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ and the following assertions hold.*

(i) The space $\mathrm{Span}\,(\mathbb{1}_A \mathrm{P} \,:\, A \in \mathcal{A}, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0))$ of simple $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$-valued functions is dense in $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$.

(ii) For any subset $E \subset L^2(\Lambda, \mathcal{A}, \|\nu\|_1)$ which is linearly dense in $L^2(\Lambda, \mathcal{A}, \|\nu\|_1)$, the space $\mathrm{Span}\,(h\mathrm{P} \,:\, h \in E, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0))$ is dense in $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$.

In some of the definitions above, it can be useful to replace $\|\nu\|_1$ can be by any σ-finite non-negative measure μ dominating $\|\nu\|_1$ and the following characterization hold (see Section 5.2 for a proof).

Proposition 3.4. *Let $(\Lambda, \mathcal{A})$ be a measurable space, $\mathcal{H}_0, \mathcal{G}_0, \mathcal{I}_0$ be three separable Hilbert spaces and ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. Let μ be a σ-finite non-negative measure dominating $\|\nu\|_1$ and set $g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$, as defined in Theorem 2.2. Then the following assertions hold.*

(a) For all $(\Phi, \Psi) \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0) \times \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{I}_0)$, (Φ, Ψ) is ν-integrable if and only if the three following assertions hold.

(i') We have $\mathrm{Im}(g^{1/2}) \subset \mathcal{D}(\Phi)$ and $\mathrm{Im}(g^{1/2}) \subset \mathcal{D}(\Psi)$, μ-a.e.

(ii') We have $\Phi g^{1/2} \in \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$ and $\Psi g^{1/2} \in \mathcal{S}_2(\mathcal{H}_0, \mathcal{I}_0)$, μ-a.e.

(iii') $(\Phi g^{1/2})(\Psi g^{1/2})^{\mathsf{H}} \in \mathcal{L}^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{G}_0, \mathcal{I}_0), \mu)$.

In this case we have

$$\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} = \int (\Phi g^{1/2})(\Psi g^{1/2})^{\mathsf{H}} \,\mathrm{d}\mu . \tag{3.5}$$

(b) For all $\Phi \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$, we have $\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ if and only if

$$\begin{cases} \mathrm{Im}(g^{1/2}) \subset \mathcal{D}(\Phi) \ \mu\text{-a.e.} \\ \Phi g^{1/2} \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu) \end{cases}$$

(c) If $\Phi, \Psi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$, then (Φ, Ψ) is ν-integrable and

$$\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} = \left[\Phi g^{1/2}, \Psi g^{1/2}\right]_{L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)} , \tag{3.6}$$

where the latter Gramian comes from Example 2.1. Hence the mapping $\Phi \mapsto \Phi g^{1/2}$ is Gramian-isometric from $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ to $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$.

3.3 Integration with respect to a random c.a.g.o.s. measure

Having all the necessary notions for a clear definition of the modular spectral domain, we now define the mapping which makes it Gramian-isometrically isomorphic to the modular time domain. This definition is often presented as a stochastic integral because it linearly and continuously maps a function to a random variable.

Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces, $(\Lambda, \mathcal{A})$ be a measurable space, and let ν be a trace-class p.o.v.m. defined on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. Given an $\mathcal{H}_0$-valued random c.a.g.o.s. measure W, we further set

$$\mathcal{H}^{W,\mathcal{G}_0} := \overline{\mathrm{Span}}^{\mathcal{G}} \left(\mathrm{P}W(A) \;:\; \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0),\ A \in \mathcal{A}\right) , \tag{3.7}$$

which is a submodule of $\mathcal{G} := \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$. As in Proposition 13 in [12, Secion 3.4] and [16, Theorem 6.9], we now define the integral of an $\mathcal{H}_0 \to \mathcal{G}_0$-operator valued valued function with respect to a random c.a.g.o.s. measure W as a Gramian-isometry from the normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$ to $\mathcal{H}^{W,\mathcal{G}_0}$. A proof can be found in Section 5.2.

Theorem 3.5. *Let $(\Lambda, \mathcal{A})$ be a measurable space and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces. Let W be an $\mathcal{H}_0$-valued random c.a.g.o.s. measure on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure ν_W. Let $\mathcal{H}^{W,\mathcal{G}_0}$ be defined as in (3.7). Then there exists a unique Gramian-isometry*

$$I_W^{\mathcal{G}_0} : \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W) \to \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$$

such that, for all $A \in \mathcal{A}$ and $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$,

$$I_W^{\mathcal{G}_0}(\mathbb{1}_A \mathrm{P}) = \mathrm{P}W(A) \quad \mathbb{P}\text{-}a.s.$$

Moreover, $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$ and $\mathcal{H}^{W,\mathcal{G}_0}$ are Gramian-isometrically isomorphic.

We can now define the integral of an operator valued function with respect to W.

Definition 3.4 (Integral with respect to a random c.a.g.o.s. measure)**.** *Under the assumptions of Theorem 3.5, we use an integral sign to denote $I_W^{\mathcal{G}_0}(\Phi)$ for $\Phi \in \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$. Namely, we write*

$$\int \Phi \, \mathrm{d}W = \int \Phi(\lambda) \, W(\mathrm{d}\lambda) := I_W^{\mathcal{G}_0}(\Phi) . \tag{3.8}$$

The following remark will be useful.

Remark 3.3. *In the setting of Definition 3.4, take $\Phi = \phi \, \mathrm{Id}_{\mathcal{H}_0}$ with $\phi : \Lambda \to \mathbb{C}$. Then, we have $\Phi \in \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0), \nu_W)$ if and only if $\phi \in L^2(\Lambda, \mathcal{A}, \|\nu_W\|_1)$. We will omit $\mathrm{Id}_{\mathcal{H}_0}$ in the notation of the integral, writing $\int \phi \, \mathrm{d}W$ for $\int \phi \mathrm{Id}_{\mathcal{H}_0} \, \mathrm{d}W$.*

We now state a straightforward result, whose proof is omitted.

Proposition 3.6. *Let $(\Lambda, \mathcal{A})$ be a measurable space, $\mathcal{H}_0$, $\mathcal{G}_0$ two separable Hilbert spaces. Let W be an $\mathcal{H}_0$-valued random c.a.g.o.s. measure on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure ν_W. Let $\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$. Then the mapping*

$$V : A \mapsto \int_A \Phi \, \mathrm{d}W = I_W^{\mathcal{G}_0}(\mathbb{1}_A \Phi) \tag{3.9}$$

is a $\mathcal{G}_0$-valued random c.a.g.o.s. measure on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure

$$\Phi \nu_W \Phi^{\mathsf{H}} : A \mapsto \int_A \Phi \mathrm{d}\nu_W \Phi^{\mathsf{H}} ,$$

which is a well defined trace-class p.o.v.m.

The c.a.g.o.s. V defined by (3.9) is said to admit the density Φ with respect to W, and we write $\mathrm{d}V = \Phi \mathrm{d}W$ (or, equivalently, $V(\mathrm{d}\lambda) = \Phi(\lambda) W(\mathrm{d}\lambda)$). In the following definition, based on Proposition 3.6, we use a signal processing terminology where Λ is seen as a set of frequencies and Φ is seen as a transfer operator function acting on the (random) input frequency distribution W.

Definition 3.5 (Filter $\hat{F}_\Phi(W)$ acting on a random c.a.g.o.s. measure in $\hat{\mathcal{S}}_\Phi$)**.** *Let $(\Lambda, \mathcal{A})$ be a measurable space, $\mathcal{H}_0$, $\mathcal{G}_0$ two separable Hilbert spaces. For a given transfer operator function $\Phi \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$, we denote by $\hat{\mathcal{S}}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$ the set of $\mathcal{H}_0$-valued random c.a.g.o.s. measures on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ whose intensity operator measures ν_W satisfy $\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$. Then, for any $W \in \hat{\mathcal{S}}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$, we say that the random $\mathcal{G}_0$-valued c.a.g.o.s. measure V defined by (3.9) is the output of the filter with transfer operator function Φ applied to the input c.a.g.o.s. measure W, and we denote $V = \hat{F}_\Phi(W)$.*

We conclude this section with a kind of Fubini theorem for interchanging a Bochner integral with a c.a.g.o.s. integral.

Proposition 3.7. *Let $(\Lambda, \mathcal{A})$ be a measurable space and $\mathcal{H}_0$, $\mathcal{G}_0$ two separable Hilbert spaces. Let W be an $\mathcal{H}_0$-valued random c.a.g.o.s. measure on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure ν_W. Let μ be a non-negative measure on a measurable space $(\Lambda', \mathcal{A}')$. Suppose that Φ is measurable from $\Lambda \times \Lambda'$ to $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and satisfies*

$$\int \left(\int \left\| \Phi(\lambda, \lambda') \right\|_{\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)} \mu(\mathrm{d}\lambda') \right)^2 \left\| \nu_W \right\|_1 (\mathrm{d}\lambda) < \infty \,, \tag{3.10}$$

$$\int \left(\int \left\| \Phi(\lambda, \lambda') \right\|^2_{\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)} \left\| \nu_W \right\|_1 (\mathrm{d}\lambda) \right)^{1/2} \mu(\mathrm{d}\lambda') < \infty \,. \tag{3.11}$$

Then we have

$$\int \left(\int \Phi(\lambda, \lambda') \, \mu(\mathrm{d}\lambda') \right) \, W(\mathrm{d}\lambda) = \int \left(\int \Phi(\lambda, \lambda') \, W(\mathrm{d}\lambda) \right) \, \mu(\mathrm{d}\lambda') \,, \tag{3.12}$$

where integrals with respect to W are as in Definition 3.4, in the left-hand side the innermost integral is understood as a Bochner integral on $L^2(\Lambda', \mathcal{A}', \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \mu)$ and in the right-hand side, the outermost integral is understood as a Bochner integral on $L^2(\Lambda', \mathcal{A}', \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P}), \mu)$.

Proof. Conditions (3.10) and (3.11) ensure that $\Phi(\lambda, \cdot) \in L^1(\Lambda', \mathcal{A}', \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \mu)$ for $\left\| \nu_W \right\|_1$-a.e. $\lambda \in \Lambda$ and that $\Phi(\cdot, \lambda') \in L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \left\| \nu \right\|_1)$ for μ-a.e. $\lambda' \in \Lambda'$, showing that the innermost integrals in both sides of (3.12) are well defined for adequate sets of λ and λ', respectively.

Let E_1 and E_2 denote the sets of functions Φ measurable from $\Lambda \times \Lambda'$ to $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and satisfying (3.10) and (3.11), respectively. We denote by $\left\| \Phi \right\|_{E_1}$ the square root of the left-hand side of (3.10) and by $\left\| \Phi \right\|_{E_2}$ the left-hand side of (3.11), which make E_1 and E_2 Banach spaces. Then, for all $\Phi \in E := E_1 \cap E_2$, concerning the left-hand side of (3.12), we have

$$\left\| \int \Phi(\cdot, \lambda') \, \mu(\mathrm{d}\lambda') \right\|^2_{\nu_W} \leq \int \left\| \int \Phi(\cdot, \lambda') \, \mu(\mathrm{d}\lambda') \right\|^2_{\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)} \mathrm{d} \left\| \nu_W \right\|_1 \leq \left\| \Phi \right\|^2_{E_1} \,,$$

as for the right-hand side, we have, setting $\mathcal{H} := \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$,

$$\int \left\| \int \Phi(\lambda, \cdot) \, W(\mathrm{d}\lambda) \right\|_{\mathcal{H}} \mathrm{d}\mu = \int \left\| \Phi(\cdot, \lambda') \right\|_{\nu_W} \mu(\mathrm{d}\lambda') \leq \left\| \Phi \right\|_{E_2} \,,$$

These two inequalities show that both sides of (3.12) seen as functions of Φ are linear continuous from E endowed with the norm $\left\| \cdot \right\|_E = \left\| \cdot \right\|_{E_1} + \left\| \cdot \right\|_{E_2}$ to $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$. Since they coincide for $\Phi(\lambda, \lambda') = 1_A(\lambda) 1_B(\lambda') \mathrm{P}$ with $A \in \mathcal{A}$, $B \in \mathcal{A}'$ and $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, this concludes the proof. ☐

4 Modular spectral domain of a weakly stationary process and applications

4.1 The Gramian-Cramér representation and general Bochner theorems

We now have all the tools to derive a spectral theory for Hilbert valued weakly-stationary processes following [12, Section 4.2]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{H}_0$ be a separable

Hilbert space and $(\mathrm{G},+)$ be a locally compact Abelian (l.c.a.) group, whose null element is denoted by 0. Recall that $\hat{\mathrm{G}}$ denotes the dual group of G defined in Section 2.1. Throughout this section we are interested in the spectral properties of a centered process valued in a separable Hilbert space and assumed to be weakly stationary in the following sense.

Definition 4.1 (Hilbert valued weakly stationary processes)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{H}_0$ be a separable Hilbert space and $(\mathrm{G},+)$ be an l.c.a. group. Then a process $X := (X_t)_{t\in\mathrm{G}}$ is said to be an $\mathcal{H}_0$-valued weakly stationary process if*

(i) For all $t \in \mathrm{G}$, $X_t \in L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$.

(ii) For all $t \in \mathrm{G}$, $\mathbb{E}[X_t] = \mathbb{E}[X_0]$. We say that X is centered if $\mathbb{E}[X_0] = 0$.

(iii) For all $t, h \in \mathrm{G}$, $\mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}(X_h, X_0)$.

(iv) The autocovariance operator function *$\Gamma_X : h \mapsto \mathrm{Cov}(X_h, X_0)$ satisfies the following continuity condition: for all $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0)$, $h \mapsto \mathrm{Tr}(\mathrm{P}\Gamma_X(h))$ is continuous on G.*

In the case of time series, $\mathrm{G} = \mathbb{Z}$, we can of course remove Condition (iv) in this definition. It is less trivial to show that, for any l.c.a. group G, we get an equivalent definition if we replace (iv) by just saying that Γ_X is continuous in w.o.t. This interesting fact is explained in the following remark in a more detailed fashion.

Remark 4.1. *The trace appearing in Assertion (iv) of Definition 4.1 is well defined for any $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0)$ and any $h \in \mathrm{G}$ since the covariance operator of variables in $L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ lies in $\mathcal{S}_1(\mathcal{H}_0)$, and the composition of a bounded operator and a trace-class operator is trace-class. Furthermore, for any $x, y \in \mathcal{H}_0$, taking $P = xy^{\mathsf{H}}$ we have $\mathrm{Tr}(\mathrm{P}\Gamma_X(h)) = \langle \Gamma_X(h)x, y\rangle_{\mathcal{H}_0}$. Hence Condition (iv) of Definition 4.1 implies the following one.*

(iv') The autocovariance operator function $\Gamma_X : h \mapsto \mathrm{Cov}(X_h, X_0)$ is continuous in w.o.t.

It is easy to find a mapping $f : \mathrm{G} \to \mathcal{S}_1(\mathcal{H}_0)$ which is continuous in w.o.t. but such that $h \mapsto \mathrm{Tr}(f(h))$ is not continuous hence does not satisfy the continuity condition imposed on Γ_X in (iv). However, it turns out that if Γ_X is the autocovariance operator function $h \mapsto \mathrm{Cov}(X_h, X_0)$ with X satisfying Conditions (i) and (iii), then Conditions (iv) and (iv') become equivalent. The reason behind this surprising fact will be made clear later in Point 1) of Remark 4.3. In other words, we can replace (iv) by (iv') without altering Definition 4.1.

As in the univariate case, the notion of weak stationarity is related to an isometric property of the lag operators, but here the covariance stationarity expressed in Condition (iii) translates into a Gramian-isometric property rather than a scalar isometric property. Namely, let $X := (X_t)_{t\in\mathrm{G}}$ satisfy Conditions (i) and (ii) and take it centered so that each X_t belongs to the normal Hilbert module $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ as defined in Example 2.2. For all $h \in \mathrm{G}$, define the lag operator of lag $h \in \mathrm{G}$ as the mapping $U_h^X : X_t \mapsto X_{t+h}$ defined for all $t \in \mathrm{G}$. Then Condition (iii) is equivalent to saying that for all $h \in \mathrm{G}$, the mapping U_h^X is Gramian-isometric on $\{X_t \,:\, t \in \mathrm{G}\}$ for the Gramian structure inherited from $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$. Thus, if this condition holds, for any lag $h \in \mathrm{G}$, using Proposition 2.3, there exists a unique Gramian-unitary operator extending U_h^X on the *modular time domain* $\mathcal{H}^X$ of X defined as the submodule of $\mathcal{H}$ generated by the X_t's, that is,

$$\mathcal{H}^X := \overline{\mathrm{Span}}^{\mathcal{H}}\left(\mathrm{P}X_t \;:\; \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0),\, t \in \mathrm{G}\right) ,$$

which is the generalization of (1.5) to a general l.c.a. group G. In fact it is convenient to introduce a slightly more general definition of the modular time domain.

Definition 4.2 ($\mathcal{G}_0$-valued modular time domain)**.** *Let $(\mathrm{G},+)$ be an l.c.a. group, and $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces. Let $X := (X_t)_{t\in\mathrm{G}}$ be a collection of variables in $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ as defined in Example 2.2. The $\mathcal{G}_0$-valued* modular time domain *of X is defined by*

$$\mathcal{H}^{X,\mathcal{G}_0} := \overline{\mathrm{Span}}^{\mathcal{M}(\Omega,\mathcal{F},\mathcal{G}_0,\mathbb{P})}\left(\mathrm{P}X_t \;:\; \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0),\, t \in \mathrm{G}\right) , \tag{4.1}$$

which is a submodule of $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$.

We now extend the (scalar) Cramér representation theorem by means of an integral with respect to a c.a.g.o.s. measure.

Theorem 4.1 (Gramian-Cramér representation theorem)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space, $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathrm{G}, +)$ be an l.c.a. group. Let $X := (X_t)_{t\in\mathrm{G}}$ be a centered weakly stationary $\mathcal{H}_0$-valued process as in Definition 4.1. Then there exists a unique regular $\mathcal{H}_0$-valued random c.a.g.o.s. measure $\hat{X}$ on $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \Omega, \mathcal{F}, \mathbb{P})$ such that*

$$X_t = \int \chi(t)\ \hat{X}(\mathrm{d}\chi) \quad \textit{for all} \quad t \in \mathrm{G}\,. \tag{4.2}$$

This result is stated in Theorem 2 in [12, Section 4.2] without the uniqueness, which appeared to be a new result in this general setting. We provide a detailed proof in Section 5.3. In fact Theorem 2 in [12, Section 4.2] contains a converse statement, which we now state separately as a lemma with its proof.

Lemma 4.2. *Let $(\mathrm{G}, +)$ be an l.c.a. group, $\mathcal{H}_0$ a separable Hilbert space and W be an $\mathcal{H}_0$-valued random c.a.g.o.s. measure on $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure ν. Define, for all $t \in \mathrm{G}$,*

$$X_t = \int \chi(t)\ W(\mathrm{d}\chi)\ .$$

Then $X = (X_t)_{t\in\mathrm{G}}$ is a centered $\mathcal{H}_0$-valued weakly stationary process with auto-covariance operator function Γ defined by

$$\Gamma(h) = \int \chi(h)\, \nu(\mathrm{d}\chi) \quad \textit{for all } h \in \mathrm{G}. \tag{4.3}$$

Proof. By Definition 3.4, $X = (X_t)_{t\in\mathrm{G}}$ is a centered $\mathcal{H}_0$-valued process satisfying (i) and (ii) in Definition 4.1. Using the Gramian-isometric property of integration with respect to W, we get for all $t, h \in \mathrm{G}$, $\mathrm{Cov}\,(X_{t+h}, X_t) = \int \chi(t+h)\overline{\chi(t)}\nu_X(\mathrm{d}\chi) = \int \chi(h)\nu_X(\mathrm{d}\chi)$ which gives (iii) in Definition 4.1 with auto-covariance operator function Γ given by (4.3). Finally, for all $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0)$, for all $h \in \mathrm{G}$, denoting by f the density of ν with respect to $\|\nu\|_1$, we have

$$\mathrm{P}\Gamma(h) = \mathrm{P} \int \chi(h)\, f(\chi)\, \|\nu\|_1(\mathrm{d}\chi) = \int \chi(h)\, \mathrm{P}\, f(\chi)\, \|\nu\|_1(\mathrm{d}\chi)\ ,$$

Since the integrand in the last integral has trace-class norm upper bounded by $\|\mathrm{P}\|_{\mathcal{L}_b(\mathcal{H}_0)}$ and $\|\nu\|_1$ is finite we get that $h \mapsto \mathrm{P}\Gamma(h)$ is continuous from $\hat{\mathrm{G}}$ to $\mathcal{S}_1(\mathcal{H}_0)$ by dominated convergence. The continuity of $h \mapsto \mathrm{Tr}(\mathrm{P}\Gamma(h))$ follows, thus showing the last point of Definition 4.1. □

With Theorem 4.1 at our disposal, we can now define the *Gramian-Cramér representation* and the *spectral operator measure* of X.

Definition 4.3 (Gramian-Cramér representation and spectral operator measure)**.** *Under the setting of Theorem 4.1, the regular c.a.g.o.s. measure $\hat{X}$ is called the* (Gramian) Cramér representation *of X and its intensity operator measure is called the* spectral operator measure *of X. It is a regular trace-class p.o.v.m. on $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{H}_0)$.*

By Lemma 4.2, we see that the auto-covariance operator function and the spectral operator measure of X are related to each other through the identity (4.3). As already hinted in the introduction, using the tools introduced in Section 3.3, we can more generally interpret the Cramér representation of Theorem 4.1 as establishing a Gramian-isometric mapping onto the modular time domain of X, starting from its *modular spectral domain* which we now introduce.

Definition 4.4 ($\mathcal{G}_0$-valued spectral time domain)**.** *Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces and $X := (X_t)_{t\in\mathrm{G}}$ be a centered weakly stationary process valued in $\mathcal{H}_0$ as in Definition 4.1. The $\mathcal{G}_0$-valued* modular spectral domain *of X is the normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module defined by*

$$\widehat{\mathcal{H}}^{X,\mathcal{G}_0} := \mathsf{L}^2(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_X)\ , \tag{4.4}$$

where ν_X is the spectral operator measure of X introduced in Definition 4.3.

We can now state that the modular time and spectral domain are Gramian-isometrically isomorphic, whose proof can be found in Section 5.3.

Theorem 4.3 (Kolmogorov isomorphism theorem)**.** *Under the setting of Theorem 4.1, for any separable Hilbert space* $\mathcal{G}_0$*, the mapping* $I_{\hat{X}}^{\mathcal{G}_0} : \Phi \mapsto \int \Phi \, \mathrm{d}\hat{X}$ *is a Gramian-unitary operator from* $\widehat{\mathcal{H}}^{X,\mathcal{G}_0}$ *to* $\mathcal{H}^{X,\mathcal{G}_0}$ *and we have* $\mathcal{H}^{X,\mathcal{G}_0} = \mathcal{H}^{\hat{X},\mathcal{G}_0}$*. Thus, the* $\mathcal{G}_0$*-valued* modular time domain $\mathcal{H}^{X,\mathcal{G}_0}$ *and the* $\mathcal{G}_0$*-valued* modular spectral domain $\widehat{\mathcal{H}}^{X,\mathcal{G}_0}$ *are* Gramian-isometrically isomorphic.

Remark 4.2. *There are two natural classes of Gramian-unitary operators respectively on the modular time and spectral domains, namely, for all* $h \in \mathrm{G}$*, the lag operator* $U_h^X : \mathcal{H}^X \to \mathcal{H}^X$ *defined as the Gramian-unitary extension of* $X_t \mapsto X_{t+h}$*,* $t \in \mathrm{G}$*, and the multiplication by* $M_h^X : \widehat{\mathcal{H}}^X \to \widehat{\mathcal{H}}^X$ *which maps* Φ *to* $\chi \mapsto \chi(h)\Phi(\chi)$*. Then, for all* $h \in \mathrm{G}$*,* U_h^X *and* M_h^X *represent the same mapping expressed either in the time domain or the spectral domain in the sense that* $U_h^X \circ I_{\hat{X}}^{\mathcal{H}_0} = I_{\hat{X}}^{\mathcal{H}_0} \circ M_h^X$*. Indeed, applying these definitions with* (4.2)*, we immediately get that* U_h^X *and* $I_{\hat{X}}^{\mathcal{H}_0} \circ M_h^X \circ \left(I_{\hat{X}}^{\mathcal{H}_0}\right)^{-1}$ *are Gramian-isometric and coincide on* $\{X_t \,:\, t \in \mathrm{G}\}$*, hence, by Proposition 2.3, coincide on* $\mathcal{H}^X$*.*

Relation (4.3) is at the core of the general Bochner theorem, which we now discuss. Recall that the *standard* (univariate) Bochner theorem can be stated as follows (see [24, Theorem 1.4.3] for existence and [24, Theorem 1.3.6] for uniqueness).

Theorem 4.4 (Bochner Theorem)**.** *Let* $(\mathrm{G}, +)$ *be an l.c.a. group and* $\gamma : \mathrm{G} \to \mathbb{C}$*. Then the two following statements are equivalent:*

(i) γ *is continuous and hermitian non-negative definite, that is, for all* $n \in \mathbb{N}$*,* $t_1, \cdots, t_n \in \mathrm{G}$ *and* $a_1, \cdots, a_n \in \mathbb{C}$*,*

$$\sum_{i,j=1}^{n} a_i \overline{a_j} \gamma(t_i - t_j) \geq 0.$$

(ii) There exists a regular finite non-negative measure ν *on* $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}))$ *such that*

$$\gamma(h) = \int \chi(h) \, \nu(\mathrm{d}\chi), \quad h \in \mathrm{G}. \tag{4.5}$$

Moreover, if Assertion (ii) holds, ν *is the unique regular non-negative measure satisfying* (4.5)*.*

There are various other ways to extend Condition (i) of Theorem 4.4 when replacing $\mathbb{C}$ by a Hilbert space $\mathcal{H}_0$.

Definition 4.5. *Let* $\mathcal{H}_0$ *be a Hilbert space and* $(\mathrm{G}, +)$ *an l.c.a. group. A function* $\Gamma : \mathrm{G} \to \mathcal{L}_b(\mathcal{H}_0)$ *is said to be*

1. *a* proper auto-covariance *operator function if* $\mathcal{H}_0$ *is separable and there exists a* $\mathcal{H}_0$*-valued weakly stationary process with autocovariance operator function* Γ*;*
2. positive definite *if for all* $n \in \mathbb{N}^*$*,* $t_1, \cdots, t_n \in \mathrm{G}$ *and* $\mathrm{P}_1, \cdots, \mathrm{P}_n \in \mathcal{L}_b(\mathcal{H}_0)$*,*

$$\sum_{i,j=1}^{n} \mathrm{P}_i \Gamma(t_i - t_j) \mathrm{P}_j^{\mathsf{H}} \succeq 0\,;$$

3. *of* positive-type *if for all* $n \in \mathbb{N}^*$*,* $t_1, \cdots, t_n \in \mathrm{G}$ *and* $x_1, \cdots, x_n \in \mathcal{H}_0$*,*

$$\sum_{i,j=1}^{n} \langle \Gamma(t_i - t_j) x_j, x_i \rangle_{\mathcal{H}_0} \geq 0\,;$$

4. hermitian non-negative definite *if for all* $n \in \mathbb{N}^*$*,* $t_1, \cdots, t_n \in \mathrm{G}$ *and* $a_1, \cdots, a_n \in \mathbb{C}$*,*

$$\sum_{i,j=1}^{n} a_i \overline{a_j} \Gamma(t_i - t_j) \succeq 0.$$

Equivalently, Γ *is hermitian non-negative definite if and only if for all* $x \in \mathcal{H}_0$*,* $t \mapsto \langle \Gamma(t)x, x \rangle_{\mathcal{H}_0}$ *is hermitian non-negative definite.*

It is straightforward to show that the definitions in Definition 4.5 are given in an increasing order of generality in the sense that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$. In the univariate case, for a continuous $\gamma : \mathrm{G} \to \mathbb{C}$ all these definitions are trivially equivalent to Assertion (i) in Theorem 4.4. A natural question for a general Hilbert space $\mathcal{H}_0$ is which definition should be used to extend the Bochner theorem. A first answer is the following corollary which is obtained as a consequence of Theorem 4.1, and the construction of $\mathcal{H}_0$-valued Gaussian c.a.g.o.s. measures of Theorem 3.1.

Corollary 4.5. *Let* $(\mathrm{G}, +)$ *be an l.c.a. group,* $\mathcal{H}_0$ *a separable Hilbert space and* $\Gamma : \mathrm{G} \to \mathcal{L}_b(\mathcal{H}_0)$*. Then the following assertions are equivalent.*

(i) The function Γ *is a proper auto-covariance operator function.*

(ii) There exists a regular trace-class p.o.v.m. ν *on* $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{H}_0)$ *such that (4.3) holds.*

Proof. The implication (i)⇒(ii) follows from Theorem 4.1. Now suppose that (ii) holds. Let W be the Gaussian c.a.g.o.s. measure with intensity operator measure ν obtained in Theorem 3.1. Then Assertion (i) follows from Lemma 4.2. □

This result extends Bochner's theorem from the point of view of $\mathcal{H}_0$-valued weakly-stationary processes so that Γ in Corollary 4.5(i) is valued in $\mathcal{S}_1(\mathcal{H}_0)$ and for all $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0)$, $h \mapsto \mathrm{Tr}(\mathrm{P}\Gamma(h))$ is continuous. It turns our that other extensions can be obtained using a purely operator theory point of view with the more general positiveness conditions of Definition 4.5. In the following theorem, $\mathcal{H}$ is not necessarily separable, Γ is not necessarily $\mathcal{S}_1(\mathcal{H})$-valued (and therefore the resulting p.o.v.m. may not be trace-class) and its continuity condition can be relaxed to continuity for the w.o.t. This result is essentially the Naimark's moment theorem of [4]. We refer to it as the *general* Bochner theorem (or general Herglotz theorem for $\mathrm{G} = \mathbb{Z}$).

Theorem 4.6 (General Bochner Theorem)**.** *Let* $(\mathrm{G}, +)$ *be an l.c.a. group,* $\mathcal{H}$ *a Hilbert space and* $\Gamma : \mathrm{G} \to \mathcal{L}_b(\mathcal{H})$*. Then the following assertions are equivalent.*

(i) Γ *is continuous in w.o.t. and positive definite.*

(ii) Γ *is continuous in w.o.t. and of positive-type.*

(iii) Γ *is continuous in w.o.t. and hermitian non-negative definite.*

(iv) There exists a regular p.o.v.m. ν *on* $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{H})$ *such that (4.3) holds.*

Moreover, if Assertion (iv) holds, ν *is the unique regular p.o.v.m. satisfying (4.3).*

It is important to note that there is a subtle difference between Assertion (ii) of Corollary 4.5 and Assertion (iv) of Theorem 4.6, namely, the latter assertion is weaker since ν is not supposed to be trace-class. In particular, we must rely on Definition 2.5 for defining the integral in (4.3) and cannot rely on the Radon-Nikodym derivative as in Theorem 2.2 (d) if ν is not trace-class.

Proof of Theorem 4.6. The equivalence between (i) and (ii) is straightforward: to show that (i)⇒(ii), take an arbitrary $x \in \mathcal{H}_0$ with unit norm and set $\mathrm{P}_i = x\, x_i^{\mathsf{H}}$ for $i = 1, \ldots, n$. To show that (ii)⇒(i), take, for any $x \in \mathcal{H}_0$, $x_i = \mathrm{P}_i^{\mathsf{H}} x$ for $i = 1, \ldots, n$. The equivalence between (ii), (iii) and (iv) is given by [4, Theorem 3]. Recall Definition 2.4 of a regular p.o.v.m.. It follows that the lastly stated fact that ν is uniquely determined by (4.3) is a consequence of the uniqueness stated in the univariate Bochner theorem (recalled in Theorem 4.4) applied to $\nu_x : A \mapsto x^{\mathsf{H}} \nu(A)\, x$ for all $x \in \mathcal{H}_0$. □

An immediate consequence of Corollary 4.5 and Theorem 4.6 is the following result.

Corollary 4.7. *Let* $(\mathrm{G}, +)$ *be an l.c.a. group,* $\mathcal{H}_0$ *a separable Hilbert space and* $\Gamma : \mathrm{G} \to \mathcal{L}_b(\mathcal{H}_0)$*. Then the following assertions are equivalent.*

(i) The function Γ *is a proper auto-covariance operator function.*

(ii) Any of the Assertions (i)–(iii) in Theorem 4.6 holds and $\Gamma(0) \in \mathcal{S}_1(\mathcal{H}_0)$*.*

Proof. By definition of the auto-covariance operator function of a weakly stationary process, it is straightforward to see that Assertion (i) implies Assertion (ii). Now, suppose that Assertion (ii) holds. By Corollary 4.5, we only need to prove Assertion (ii) of Corollary 4.5, which is what we almost get in Assertion (iv) of Theorem 4.6, except that we have to prove that, additionally, ν is trace-class. Applying (4.3) with $h = 0$, we get that $\nu(\hat{\mathrm{G}}) = \Gamma(0)$, which is assumed to be in $\mathcal{S}_1(\mathcal{H}_0)$ in the present Assertion (ii). Thus by Lemma 2.1, ν is indeed trace-class and the proof is concluded. □

Remark 4.3. *Let us briefly comment on the equivalence established in Corollary 4.7.*

1) *In Condition (iv) of Definition 4.1, we required a condition on Γ which is stronger than continuity in w.o.t. However in Assertion (ii) of Corollary 4.7, the continuity of Γ is only needed in the w.o.t. This means that we can replace the continuity Condition (iv) in Definition 4.1 by continuity in w.o.t. as in Remark 4.1 (iv') without changing the overall definition of a weakly stationary process.*

2) *The previous remark is related to a fact established in Proposition 3 of [12, Section 4.2], which states the equivalence between being* scalar stationary *and being* operator stationary*. The latter definition is the same as our Definition 4.1, and the former one amounts to replace Condition (iv) in Definition 4.1 by assuming that for all $x \in \mathcal{H}_0$, $x^{\mathsf{H}}\Gamma x : h \mapsto x^{\mathsf{H}}\Gamma(h)x$ is continuous and hermitian non-negative definite. But this amounts to says that Γ itself is continuous in the w.o.t. and hermitian non-negative definite. Since $\Gamma(0) \in \mathcal{S}_1(\mathcal{H}_0)$ is a consequence of Assertion (i) in Definition 4.1, Corollary 4.7 indeed implies the equivalence established in Proposition 3 of [12, Section 4.2].*

4.2 Composition and inversion of filters

With the construction of the spectral theory for weakly-stationary processes of Section 4.1, the study of linear filters for such processes is easily derived. Indeed, we are now able to give the most general definition of linear filtering, characterize the spectral structure of the filtered process and provide results on compositions and inversion of linear filters. Then, in the next section, we will provide a general statement of harmonic principal component analysis for weakly stationary processes valued in a separable Hilbert space.

Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces. Consider a linear lag-invariant filter with input an $\mathcal{H}_0$-valued weakly stationary stochastic process $X = (X_t)_{t\in\mathrm{G}}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and output a $\mathcal{H}^{X,\mathcal{G}_0}$-valued process $Y = (Y_t)_{t\in\mathrm{G}}$. Then, for all $t \in \mathrm{G}$, $Y_t = U_t^X Y_0$, which, by Remark 4.2, reads, in the spectral domain,

$$Y_t = \int \chi(t)\, \Phi(\chi)\, \hat{X}(\mathrm{d}\chi)\;, \qquad t \in \mathrm{G}\;,$$

where Φ is called the *transfer operator function* of the filter. Therefore, the output $Y = (Y_t)_{t\in\mathrm{G}}$ is well defined in $\mathcal{H}^X$ if and only if

$$\hat{X} \in \hat{\mathcal{S}}_\Phi(\Omega, \mathcal{F}, \mathbb{P}) \quad \text{or, equivalently,} \quad \Phi \in \widehat{\mathcal{H}}^{X,\mathcal{G}_0}\;, \tag{4.6}$$

where $\hat{\mathcal{S}}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$ is as in Definition 3.5 and $\widehat{\mathcal{H}}^{X,\mathcal{G}_0}$ denotes the modular spectral domain of Definition 4.4. Then the output $Y = (Y_t)_{t\in\mathrm{G}}$ is equivalently defined by its spectral random c.a.g.o.s. measure $\hat{Y} = \hat{F}_\Phi(\hat{X})$. For convenience we write, in the time domain,

$$X \in \mathcal{S}_\Phi(\Omega, \mathcal{F}, \mathbb{P}) \quad \text{and} \quad Y = F_\Phi(X)\;, \tag{4.7}$$

for the assertions $\hat{X} \in \hat{\mathcal{S}}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$ and $\hat{Y} = \hat{F}_\Phi(\hat{X})$.

Many examples in the literature rely on a *time-domain* description of the filtering obtained as in the following example.

Example 4.1 (Convolutional filtering)**.** *Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces. Let $X = (X_t)_{t\in\mathrm{G}}$ be an $\mathcal{H}_0$-valued weakly stationary stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let η be the Haar measure on G (see [24, Chapter 1]) and $\Phi \in L^1(\mathrm{G}, \mathcal{B}(\mathrm{G}), \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \eta)$. Define the process $Y = (Y_t)_{t\in\mathrm{G}}$ by the* time domain convolutional filtering

$$Y_t = \int \Phi(s)\, X_{t-s}\, \eta(\mathrm{d}s)\;, \quad t \in \mathrm{G}\;,$$

where the integral is a Bochner integral on $L^1(\mathrm{G}, \mathcal{B}(\mathrm{G}), \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P}), \eta)$. Then, using Proposition 3.7 and defining $\hat{\Phi} : \hat{\mathrm{G}} \to \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ by the following Bochner integral on $L^1(\mathrm{G}, \mathcal{B}(\mathrm{G}), \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \eta)$,

$$\hat{\Phi}(\chi) = \int \Phi(s)\, \overline{\chi(s)}\, \eta(\mathrm{d}s)\;,$$

it is straightforward to show that $\hat{\Phi} \in \mathsf{L}^2(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_X)$ and $Y = F_{\hat{\Phi}}(X)$.

We have the following result on the composition and inversion of general filters, which relies on the Gramian-isometric relationships of Definition 2.11. Its proof can be found in Section 5.4.

Proposition 4.8 (Composition and inversion of filters on weakly stationary time series)**.** *Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces and pick a transfer operator function $\Phi \in \mathbb{F}_{\mathcal{O}}\left(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{H}_0, \mathcal{G}_0\right)$. Let X be a centered weakly stationary $\mathcal{H}_0$-valued process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with spectral operator measure ν_X. Suppose that $X \in \mathcal{S}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$ and set $Y = F_\Phi(X)$, as defined in (4.7). Then the three following assertions hold.*

(i) For any separable Hilbert space $\mathcal{I}_0$, we have $\mathcal{H}^{Y,\mathcal{I}_0} \underset{\approx}{\subsetneq} \mathcal{H}^{X,\mathcal{I}_0}$.

(ii) For any separable Hilbert space $\mathcal{I}_0$ and all $\Psi \in \mathbb{F}_{\mathcal{O}}\left(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{G}_0, \mathcal{I}_0\right)$, we have $X \in \mathcal{S}_{\Psi\Phi}(\Omega, \mathcal{F}, \mathbb{P})$ if and only if $F_\Phi(X) \in \mathcal{S}_\Psi(\Omega, \mathcal{F}, \mathbb{P})$, and in this case, we have

$$F_\Psi \circ F_\Phi(X) = F_{\Psi\Phi}(X). \tag{4.8}$$

(iii) Suppose that Φ is injective $\|\nu_X\|_1$-a.e. Then $X = F_{\Phi^{-1}} \circ F_\Phi(X)$, where we define $\Phi^{-1}(\lambda) := \left(\Phi(\lambda)_{|\mathcal{D}(\Phi(\lambda)) \to \mathrm{Im}(\Phi(\lambda))}\right)^{-1}$ with domain $\mathrm{Im}(\Phi(\lambda))$ for all $\lambda \in \{\Phi$ is injective$\}$ and $\Phi^{-1}(\lambda) = 0$ otherwise. Moreover, Assertion (i) above holds with $\underset{\approx}{\subsetneq}$ replaced by $\cong$.

4.3 Cramér-Karhunen-Loève decomposition

Let $\mathcal{H}_0$ be a separable Hilbert space with (possibly infinite) dimension N and $X = (X_t)_{t \in \mathrm{G}}$ be a centered, $\mathcal{H}_0$-valued weakly-stationary process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Cramér representation $\hat{X}$ and spectral operator measure ν_X.

The Cramér-Karhunen-Loève decomposition amounts to give a rigorous meaning to the formula

$$\hat{X}(\mathrm{d}\chi) = \sum_{0 \le n < N} \phi_n(\chi) \otimes \phi_n(\chi)\, \hat{X}(\mathrm{d}\chi)\ , \tag{4.9}$$

where, for all $\chi \in \hat{\mathrm{G}}$, $(\phi_n(\chi))_{0 \le n < N}$ is an orthonormal sequence in $\mathcal{H}_0$ chosen in such a way that the summands in (4.9) are uncorrelated and the notation $\otimes$ is recalled in Section 2. Such a decomposition provides a way to derive the harmonic principal component analysis of the process X, which is an approximation of X by a finite rank linear filtering. In recent works, the functional Cramér-Karhunen-Loève decomposition is achieved under additional assumptions on ν_X such as having a continuous density with respect to the Lebesgue measure (in [26]) or at most finitely many atoms (in [30]). In fact, thanks to the Radon-Nikodym property of trace-class p.o.v.m.'s of Theorem 2.2, there is no need for such additional assumptions. Instead, we rely on the following lemma, whose proof can be found in Section 5.5.

Lemma 4.9 (Eigendecomposition of a trace-class p.o.v.m.)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space with dimension $N \in \{1, \dots, +\infty\}$. Let ν be a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ and μ a σ-finite dominating measure of ν, e.g. its variation norm $\|\nu\|_1$. Then there exist sequences $(\sigma_n)_{0 \le n < N}$ and $(\phi_n)_{0 \le n < N}$ of $(\Lambda, \mathcal{A}) \to (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ and $(\Lambda, \mathcal{A}) \to (\mathcal{H}_0, \mathcal{B}(\mathcal{H}_0))$ measurable functions, respectively, such that the following assertions hold.*

(i) For all $\lambda \in \Lambda$, $(\sigma_n(\lambda))_{0 \le n < N}$ is non-increasing and $\sum_{0 \le n < N} \sigma_n(\lambda) < \infty$.

(ii) For all $\lambda \in \Lambda$, $(\phi_n(\lambda))_{0 \le n < N}$ is orthonormal.

(iii) The trace-class p.o.v.m. ν admits the density

$$f : \lambda \mapsto \sum_{0 \le n < N} \sigma_n(\lambda)\, \phi_n(\lambda) \otimes \phi_n(\lambda)\ ,$$

with respect to μ, where the convergence holds absolutely in $\mathcal{S}_1$ for each $\lambda \in \Lambda$.

Moreover, using the notations $\phi_n^{\mathsf{H}} : \lambda \mapsto \phi_n(\lambda)^{\mathsf{H}}$ and $\phi_n \otimes \phi_n : \lambda \mapsto \phi_n(\lambda) \otimes \phi_n(\lambda)$, we have the following properties.

(iv) The sequence $(\phi_n^{\mathsf{H}})_{0 \le n < N}$ is orthogonal in $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathbb{C}), \nu)$.

(v) The sequence $(\phi_n \otimes \phi_n)_{0 \le n < N}$ is Gramian-orthogonal in $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0), \nu)$.

(vi) *The* $\mathcal{L}_b(\mathcal{H}_0)$*-valued mapping* $\sum_{0\leq n<N} \phi_n \otimes \phi_n$ *is equal to the mapping* $\lambda \mapsto \mathrm{Id}_{\mathcal{H}_0}$ *in* $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0), \nu)$.

We have the following remark about Assertion (vi).

Remark 4.4. *By Assertions (i)-(iii), for all* $\lambda \in \Lambda$, $\sum_{0\leq n<N} \phi_n(\lambda) \otimes \phi_n(\lambda)$ *is the orthogonal projection onto the closure of the range of* $f(\lambda)$*. Thus, Assertion (vi) says that this projection is equal to* $\mathrm{Id}_{\mathcal{H}_0}$ *in* $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0), \nu)$*. It is not equivalent to saying that* $\sum_{0\leq n<N} \phi_n \otimes \phi_n = \mathrm{Id}_{\mathcal{H}_0}$, $\|\nu\|_1$*-a.e. since it may happen that the range of* $f(\lambda)$ *is dense in* $\mathcal{H}_0$ *for none of the* λ*'s, in which case we have Assertion (vi) at the same time as* $\{\sum_{0\leq n<N} \phi_n \otimes \phi_n = \mathrm{Id}_{\mathcal{H}_0}\} = \emptyset$.

Applying Lemma 4.9 to the trace-class p.o.v.m. ν_X, we deduce that

$$\hat{X} = \hat{F}_{\left(\sum_{0\leq n<N} \phi_n\otimes\phi_n\right)}(\hat{X}) = \sum_{0\leq n<N} \hat{F}_{\phi_n\otimes\phi_n}(\hat{X}) \; , \tag{4.10}$$

where $(\hat{F}_{\phi_n\otimes\phi_n}(\hat{X}))_{0\leq n<N}$ are uncorrelated random c.a.g.o.s.'s on $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{H}_0)$. In other words, (4.9) holds both with $\hat{X}$ in the sum sign or out of it in the right-hand side.

Note that for all $n \in \mathbb{N}$, $\hat{F}_{\phi_n\otimes\phi_n}(\hat{X}) = \hat{F}_{\phi_n} \circ \hat{F}_{\phi_n^{\mathsf{H}}}(\hat{X})$ and that by (iv) of Lemma 4.9, $(\hat{F}_{\phi_n^{\mathsf{H}}}(\hat{X}))_{0\leq n<N}$ is a sequence of uncorrelated $\mathbb{C}$-valued c.a.o.s. measures. Hence, interpreting (4.10) in the time domain, we get a decomposition of the process $X = (X_t)_{t\in\mathrm{G}}$ based on a collection of the uncorrelated univariate processes $(F_{\phi_n^{\mathsf{H}}}(X))_{0\leq n<N}$.

The following general formulation of a harmonic principal components analysis for $\mathcal{H}_0$-valued weakly-stationary processes then follows.

Proposition 4.10 (Harmonic functional principal components analysis)**.** *Let* $\mathcal{H}_0$ *be a separable Hilbert space and* $X = (X_t)_{t\in\mathrm{G}}$ *be a centered,* $\mathcal{H}_0$*-valued weakly-stationary process defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with spectral operator measure* ν_X*. Let* $(\sigma_n)_{0\leq n<N}$ *and* $(\phi_n)_{0\leq n<N}$ *be given as in Lemma 4.9 for some dominating measure* μ *of* ν_X*, for instance* $\mu = \|\nu_X\|_1$*. Let* $q : \hat{\mathrm{G}} \to \mathbb{N}^*$ *be a measurable function. Then for all* $t \in \mathrm{G}$,

$$\min\left\{\mathbb{E}\left[\left\|X_t - [F_\Theta(X)]_t\right\|^2\right] \;:\; \Theta \in \mathsf{L}^2(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{O}(\mathcal{H}_0), \nu_X),\ \mathrm{rank}(\Theta) \leq q\right\}$$

is equal to

$$\int_{\hat{\mathrm{G}}} \sum_{q(\chi)\wedge N\leq n<N} \sigma_n(\chi)\, \mu(\mathrm{d}\chi) \; ,$$

and the minimum is achieved for

$$\Theta : \chi \mapsto \sum_{0\leq n<q(\chi)\wedge N} \phi_n(\chi) \otimes \phi_n(\chi) \; .$$

Proof. Let

$$f_X(\chi) = \sum_{0\leq n<N} \sigma_n(\chi)\, \phi_n(\chi) \otimes \phi_n(\chi)$$

denotes the density of ν_X with respect to μ as given by Lemma 4.9. We have, for all $t \in \mathrm{G}$ and $\Theta \in \mathsf{L}^2(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{O}(\mathcal{H}_0), \nu_X)$,

$$[F_\Theta(X)]_t = \int \chi(t)\, \Theta(\chi)\, \hat{X}(\mathrm{d}\chi) \; ,$$

and thus by isometric isomorphism between the spectral domain and the time domain,

$$\mathbb{E}\left[\|X_t - [F_\Theta(X)]_t\|^2\right] = \int \left\|(\mathrm{Id}_{\mathcal{H}_0} - \Theta(\chi)) f_X^{1/2}(\chi)\right\|^2_{\mathcal{S}_2(\mathcal{H}_0)} \mu(\mathrm{d}\chi) \; .$$

The result is then obtained by observing that, for each $\chi \in \hat{\mathrm{G}}$, the norm in the integral is minimal under the constraint $\mathrm{rank}(\Theta(\chi)) \leq q(\chi)$ for $\Theta(\chi) = \sum_{0\leq n<q(\chi)\wedge N} \phi_n(\chi) \otimes \phi_n(\chi)$. ☐

4.4 Comparison with recent approaches

We can now provide a more thorough comparison with the recent works establishing a spectral theory for functional time series mentioned in the introduction. Hence we take $\mathrm{G} = \mathbb{Z}$ in this section, so that $\chi \in \hat{\mathrm{G}}$ can be replaced by $\lambda \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ (or $(-\pi, \pi]$) with $\chi(h)$ replaced by $\mathrm{e}^{\mathrm{i}\lambda h}$. The functional case usually corresponds to setting $\mathcal{H}_0 = L^2(0, 1)$ but this is unimportant for the following discussion.

As hinted in the introduction, the major benefit of the construction developed in the previous sections is to clarify the spectral domain of a functional weakly-stationary process X as being defined as a set of operator-valued functions, namely $\mathsf{L}^2(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_X)$. Moreover, the Gramian-Cramér representation, as stated in Theorem 4.1, is a particular instance of the Gramian-unitary operator between the spectral domain and the time domain, based on the integral of Definition 3.4. In contrast, in [30] the isometric isomorphism stated in their Theorem 4.4 is similar to the one expressed in [11] and recalled in the introduction as the isometric extension of (1.2) to $H_X = \overline{\mathrm{Span}}\,(X_t\,,\ t \in \mathbb{Z})$. Moreover, in recent approaches, $\hat{X}$ in (1.2) is defined as a *functional orthogonal increment process* and the integral is referred to as a Riemann–Stieltjes integral with respect to $\hat{X}$. This notion, used for the Cramér representations exhibited in [19, 20, 26, 29, 30], and which follows the construction of [23] for univariate weakly stationary time series, is based on the following definition.

Definition 4.6 (Functional orthogonal increment processes)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space. A random process $(Z_\lambda)_{\lambda \in [-\pi,\pi]}$ valued in $\mathcal{H}_0$ is said to be a functional orthogonal increment process if the three following assertions hold.*

(i) *We have $Z_{-\pi} = 0$ a.s. and, for all $\lambda \in (-\pi, \pi]$, $Z_\lambda \in \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ (as defined in Example 2.2).*

(ii) *For all $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [-\pi, \pi]$, with $\lambda_2 \geq \lambda_1$ and $\lambda_4 \geq \lambda_3$, we have*

$$(\lambda_1, \lambda_2] \cap (\lambda_3, \lambda_4] = \emptyset \Rightarrow \mathrm{Cov}\,(Z_{\lambda_4} - Z_{\lambda_3}, Z_{\lambda_2} - Z_{\lambda_2}) = 0\ .$$

(iii) *For all $\lambda \in [-\pi, \pi]$, $\lim_{\epsilon \downarrow 0} \mathbb{E}\left[\|Z_{\lambda+\epsilon} - Z_\lambda\|^2_{\mathcal{H}_0}\right] = 0$.*

Of course, Definition 4.6 can be related to random c.a.g.o.s. measures as in Definition 3.2 with $\Lambda = (-\pi, \pi]$ and $\mathcal{A} = \mathcal{B}((-\pi, \pi])$. Indeed, it is straightforward to show that, if W is an $\mathcal{H}_0$-valued random c.a.g.o.s. measure on $((-\pi, \pi], \mathcal{B}((-\pi, \pi]), \Omega, \mathcal{F}, \mathbb{P})$, then setting

$$Z_{-\pi} = 0 \quad \text{and} \quad Z_\lambda = W((-\pi, \lambda])\,, \quad \lambda \in (-\pi, \pi]\,, \tag{4.11}$$

we get a Gramian-orthogonal increment process. Now, conversely, if $(Z_\lambda)_{\lambda \in [-\pi,\pi]}$ is a Gramian-orthogonal increment process, then there exists a unique $\mathcal{H}_0$-valued random c.a.g.o.s. W on $((-\pi, \pi], \mathcal{B}((-\pi, \pi]), \Omega, \mathcal{F}, \mathbb{P})$ such that (4.11) holds. This will be done in Proposition 5.3.

We conclude this discussion by comparing Theorem 4.6 to the functional Herglotz theorem [30, Theorem 3.7]. In the following discussion, (i)–(iv) all refer to the assertions in Theorem 4.6. First note that p.o.v.m.'s are equivalent to the notion of operator-valued measures defined in [30] up to an isomorphism between monoids. Hence Theorem 3.7 in [30] is equivalent to stating the equivalence between Assertions (ii) and (iv). However the proof of Theorem 3.7 in [30] is different from the proof of the equivalence (ii)⇔(iv) proposed in [4]. A closer look at the literature in operator theory shows that the implication (iii) ⇒ (iv) appear commonly as an ingredient of the proof of Stone's theorem, see *e.g.* [2, 25], [10, §VI and VII]. Since (ii) obviously implies (iii), this indicates that the implication (ii) ⇒ (iv) is a classical result. In contrast, it seems that little attention has been given to the converse implication (iv)⇒(ii). The proof of this implication is included in the proof of [4, Theorem 2]. Berberian claims there that *"[He does] not know how to prove [it] without using dilation theory"*. The proof of the same implication given in [30] relies on the computation of $\int \langle \nu(\mathrm{d}\chi)x(\chi), x(\chi)\rangle$ where ν is an operator-valued measure in the sense of their Definition 3.5, see [30, Lines 3 and 4, Page 3695]. However the rigorous definition of such an integral is unclear to us in their context. For sake of completeness, we provide a simple proof of (iv)⇒(ii) in Section 5.3.

5 Postponed proofs

5.1 Proofs of Section 2

We start with two useful lemmas, in which $(\Lambda, \mathcal{A})$ is a measurable space, $\mathcal{H}_0, \mathcal{G}_0$ are two separable Hilbert spaces and μ is a non-negative measure on $(\Lambda, \mathcal{A})$.

Lemma 5.1. *Let $\mathcal{E} = \mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$ or $\mathcal{S}_p(\mathcal{H}_0, \mathcal{G}_0)$ where $p \in \{1, 2\}$ and. Then a function $\Phi : \Lambda \to \mathcal{E}$ is measurable if and only if it is simply measurable.*

Proof. By (2.1), we only need to show that, if Φ is simply measurable then it is measurable. The space $\mathcal{E}$ is separable because the set of finite rank operators is dense in $\mathcal{E}$ for the norm $\|\cdot\|$ if $\mathcal{E} = \mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$ and $\|\cdot\|_p$ if $\mathcal{E} = \mathcal{S}_p(\mathcal{H}_0, \mathcal{G}_0)$. By Pettis's measurability theorem, this implies that it is enough to show that for all $f \in \mathcal{E}^*$, $f \circ \Phi$ is a measurable complex-valued function. By [7, Theorems 19.1, 18.14, 19.2] we get that $\mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)^*$, $\mathcal{S}_1(\mathcal{H}_0, \mathcal{G}_0)^*$ and $\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)^*$ are respectively isometrically isomorphic to $\mathcal{S}_1(\mathcal{H}_0, \mathcal{G}_0)$, $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and $\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$ and the duality relation can be defined on $\mathcal{E} \times \mathcal{E}^*$ as $(\mathrm{P}, \mathrm{Q}) \mapsto \mathrm{Tr}(\mathrm{Q}^{\mathsf{H}}\mathrm{P})$. This means that we only have to show measurability of the complex-valued functions $\lambda \mapsto \mathrm{Tr}(\mathrm{P}^{\mathsf{H}}\Phi(\lambda))$ for all $\mathrm{P} \in \mathcal{E}^*$. Let $(\phi_k)_{k\in\mathbb{N}}, (\psi_k)_{k\in\mathbb{N}}$ be Hilbert basis of $\mathcal{H}_0$ and $\mathcal{G}_0$ respectively, then $\mathrm{Tr}(\mathrm{P}^{\mathsf{H}}\Phi(\lambda)) = \sum_{k\in\mathbb{N}} \langle \Phi(\lambda)\phi_k, \mathrm{P}\psi_k \rangle_{\mathcal{G}_0}$ which defines a measurable function of λ by simple measurability of Φ. □

Lemma 5.2. *Let $\Phi \in L^1(\Lambda, \mathcal{A}, \mathcal{S}_1^+(\mathcal{H}_0), \mu)$ and define the function $\Phi^{1/2} : \lambda \mapsto \Phi(\lambda)^{1/2}$. Then $\Phi^{1/2} \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0), \mu)$.*

Proof. Simple measurability of $\Phi^{1/2}$ is given by Lemma 2 in [12, Section 3.4] and therefore, by Lemma 5.1, $\Phi^{1/2} \in \mathbb{F}(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0))$. The fact that $\Phi^{1/2} \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0), \mu)$ then follows from the identity $\left\|\Phi^{1/2}(\lambda)\right\|_2^2 = \|\Phi(\lambda)\|_1$. □

We now provide the proofs of Lemma 2.1 and Theorem 2.2.

Proof of Lemma 2.1. The first point comes from the fact that for all $A \in \mathcal{A}$, $\nu(A) \preceq \nu(\Lambda)$. Now, if ν is trace-class, then (2.2) is easily verified for the norm $\|\cdot\|_1$ using the fact that $\|\cdot\|_1 = \mathrm{Tr}(\cdot)$ for positive operators. Finally, by definition of $\|\nu\|_1$, regularity of $\|\nu\|_1$ is equivalent to regularity of ν as an $\mathcal{S}_1(\mathcal{H}_0)$-valued measure which clearly implies regularity of $\nu_x = x^{\mathsf{H}}\nu(\cdot)x$ for all $x \in \mathcal{H}_0$. Suppose now that for all $x \in \mathcal{H}_0$, ν_x is regular, then let $(e_k)_{k\in\mathbb{N}}$ be a Hilbert basis of $\mathcal{H}_0$, and define for all $n \in \mathbb{N}$, the non-negative measure $\mu_n := \sum_{k=0}^n \nu_{e_k}$ such that for all $A \in \mathcal{A}$, $\|\nu\|_1(A) = \lim_{n\to+\infty} \mu_n(A) = \sup_{n\in\mathbb{N}} \mu_n(A)$. Then, by Vitali-Hahn-Sakh-Nikodym's theorem (see [5]), the sequence $(\mu_n)_{n\in\mathbb{N}}$ is uniformly countably additive which implies regularity of $\|\nu\|_1$ by Lemma 23 in [8, Chapter VI, Section 2]. □

Proof of Theorem 2.2. The general case where μ is σ-finite follows from the finite case. We therefore assume that μ is finite in the following. Suppose that $\|\nu\|_1 \ll \mu$. Then, since $\mathcal{H}_0$ is separable, $\mathcal{S}_1(\mathcal{H}_0)$ is the dual of the separable space $\mathcal{K}(\mathcal{H}_0)$. It is thus a *separable dual space* and Theorem 1 in [8, Chapter III, Section 3] gives the existence and uniqueness of a density $g \in L^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{H}_0), \mu)$ satisfying (2.3). Then for all $x \in \mathcal{H}_0$ and $A \in \mathcal{A}$,

$$\int_A \langle g(\lambda)x, x\rangle_{\mathcal{H}_0} \, \mu(\mathrm{d}\lambda) = \langle \nu(A)x, x\rangle_{\mathcal{H}_0} \geq 0 \, ,$$

and there exists a set $A_x \in \mathcal{A}$ with $\mu(A_x^c) = 0$ and $\langle g(\lambda)x, x\rangle_{\mathcal{H}_0} \geq 0$ for all $\lambda \in A_x$. Taking $(x_n)_{n\in\mathbb{N}}$ a dense countable subset of $\mathcal{H}_0$ we get that $g \in \mathcal{S}_1^+(\mathcal{H}_0)$ on $A = \bigcap_{n\in\mathbb{N}} A_{x_n}$ thus proving Assertion (a). Assertion (b) then follows from Lemma 5.2. Moreover, taking the trace in (2.3) gives for all $A \in \mathcal{A}$,

$$\|\nu\|_1(A) = \int_A \|g\|_1 \, \mathrm{d}\mu$$

which gives Assertion (c). Assertion (d) is easy to get by extending the case $f = 1_A$ for $A \in \mathcal{A}$ to simple functions and then using the density of simple functions. □

5.2 Proofs of Section 3

It is in fact better to start with the following proof because Theorem 3.2 basically follows from Proposition 3.4.

Proof of Proposition 3.4. Let $f = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$ as in Theorem 2.2. Using that $\|\nu\|_1(\{g=0\}) = \int_{\{g=0\}} \|g\|_1 \, \mathrm{d}\mu = 0$ and $g = f\|g\|_1$ μ-a.e. by uniqueness of the density, we get that

$$\|g\|_1 > 0 \quad \|\nu\|_1\text{-a.e.} \quad \text{and} \quad g = f\|g\|_1 \quad \mu\text{-a.e.} \tag{5.1}$$

(and thus also $\|\nu\|_1$-a.e. since $\|\nu\|_1 \ll \mu$). Assertion (a) follows easily. Let us for instance detail the proof of the equivalence between (i') and (i) of Definition 3.3. The left-hand side of (5.1) gives that

$$\|\nu\|_1 \left(\left\{\mathrm{Im}(f^{1/2}) \not\subset \mathcal{D}(\Phi)\right\}\right) = \|\nu\|_1 \left(\left\{\mathrm{Im}(f^{1/2}) \not\subset \mathcal{D}(\Phi)\right\} \cap \{g \neq 0\}\right) , \tag{5.2}$$

and its right-hand side yields

$$\begin{aligned} \mu \left(\left\{\mathrm{Im}(f^{1/2}) \not\subset \mathcal{D}(\Phi)\right\} \cap \{g \neq 0\}\right) &= \mu \left(\left\{\mathrm{Im}(g^{1/2}) \not\subset \mathcal{D}(\Phi)\right\} \cap \{g \neq 0\}\right) \\ &= \mu \left(\left\{\mathrm{Im}(g^{1/2}) \not\subset \mathcal{D}(\Phi)\right\}\right) , \end{aligned} \tag{5.3}$$

since $\left\{\mathrm{Im}(g^{1/2}) \not\subset \mathcal{D}(\Phi)\right\} \cap \{g = 0\} = \emptyset$. To get (i') $\Leftrightarrow$ (i), we note that

$$\|\nu\|_1 \left(\left\{\mathrm{Im}(f^{1/2}) \not\subset \mathcal{D}(\Phi)\right\} \cap \{g \neq 0\}\right) = \int_{\left\{\mathrm{Im}(f^{1/2}) \not\subset \mathcal{D}(\Phi)\right\} \cap \{g \neq 0\}} \|g\|_1 \, \mathrm{d}\mu ,$$

and thus the right-hand side of (5.2) is zero if and only if the left-hand side of (5.3) is. Equivalences (ii) $\Leftrightarrow$ (ii') and (iii) $\Leftrightarrow$ (iii') and Relation (3.5) are easy consequences of (5.1). Assertions (b) and (c) come easily using the definition of $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$. Measurability of $\Phi g^{1/2}$ and $(\Phi g^{1/2})(\Phi g^{1/2})$ are ensured by $\mathcal{O}$-measurability of Φ, simple measurability of f and Lemma 5.1. $\square$

We can now derive Theorem 3.2.

Proof of Theorem 3.2. All theses results are easily derived from Proposition 3.4 and the module nature of $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$. The only difficulty lies in showing the completeness of $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$, which is detailed in the proof of Theorem 11 of [12, Section 3.4]. $\square$

Proof of Theorem 3.3. In the first two steps of the proof of Theorem 12 in [12, Section 3.4] (see also [16, Theorem 4.22]), it is shown that, if $\Phi \in \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ and $\epsilon > 0$, there exists $\Psi \in L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1) \subset \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ such that $\|\Phi - \Psi\|_\nu < \epsilon$. This implies that $L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ is dense in $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$. Then Assertion (i) follows using (2.5) and the usual density of simple functions. Assertion (ii) then follows by approximating, for any $A \in \mathcal{A}$ and $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ the function $1_A\mathrm{P}$ by $g\mathrm{P}$ with $g \in \mathrm{Span}\,(E)$ arbitrarily close to 1_A in $L^2(\Lambda, \mathcal{A}, \|\nu\|_1)$. $\square$

Proof of Theorem 3.5. We set $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ and $\mathcal{G} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$. For all $A, B \in \mathcal{A}$ and $\mathrm{P}, \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, we have, by Theorem 3.2,

$$\begin{aligned} [1_A\mathrm{P}, 1_B\mathrm{Q}]_{\nu_W} &= \mathrm{P}\nu_W(A \cap B)\mathrm{Q}^{\mathsf{H}} \\ &= \mathrm{P}\,\mathrm{Cov}\,(W(A), W(B))\,\mathrm{Q}^{\mathsf{H}} \\ &= \mathrm{Cov}\,(\mathrm{P}W(A), \mathrm{Q}W(B)) \\ &= [\mathrm{P}W(A), \mathrm{Q}W(B)]_{\mathcal{G}} . \end{aligned}$$

Then Proposition 2.3, applied to $J = \mathcal{A} \times \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ with $v_{(A,\mathrm{P})} = 1_A\mathrm{P}$ and $w_{(A,\mathrm{P})} = \mathrm{P}W(A)$, gives that there exists a unique Gramian-isometric operator

$$I_W^{\mathcal{G}_0} : \overline{\mathrm{Span}}^{\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)} \left(1_A\mathrm{QP} \;:\; A \in \mathcal{A}, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \mathrm{Q} \in \mathcal{L}_b(\mathcal{G}_0)\right) \to \mathcal{G} \tag{5.4}$$

such that for all $A \in \mathcal{A}, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, $I_W^{\mathcal{G}}(1_A\mathrm{P}) = \mathrm{P}W(A)$ and, in addition,

$$\mathrm{Im}(I_W^{\mathcal{G}_0}) = \overline{\mathrm{Span}}^{\mathcal{G}} \left(\mathrm{QP}W(A) \;:\; A \in \mathcal{A}, \mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \mathrm{Q} \in \mathcal{L}_b(\mathcal{G}_0)\right) . \tag{5.5}$$

Now, note that

$$\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0)=\{\mathrm{QP}\;:\;\mathrm{P}\in\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\mathrm{Q}\in\mathcal{L}_b(\mathcal{G}_0)\}\;. \tag{5.6}$$

This gives that

$$\mathrm{Span}\left(1_A\mathrm{QP}\;:\;A\in\mathcal{A},\mathrm{P}\in\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\mathrm{Q}\in\mathcal{L}_b(\mathcal{G}_0)\right)=\mathrm{Span}\left(1_A\mathrm{P}\;:\;A\in\mathcal{A},\mathrm{P}\in\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0)\right)\;.$$

Therefore, by Theorem 3.3, the domain of $I_W^{\mathcal{G}_0}$ in (5.4) is the whole space $\mathsf{L}^2(\Lambda,\mathcal{A},\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu_W)$. Finally, (5.6) with (5.5) yields

$$\mathrm{Im}(I_W^{\mathcal{G}_0})=\overline{\mathrm{Span}}^{\mathcal{G}}\left(\mathrm{P}W(A)\;:\;A\in\mathcal{A},\mathrm{P}\in\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0)\right)=\mathcal{H}^{W,\mathcal{G}_0}\;,$$

which concludes the proof. □

We conclude this section with a useful result for comparing random c.a.g.o.s. measures as introduced in Section 3.1 and Gramian-orthogonal increment processes as in Definition 4.6.

Proposition 5.3. *Let $(Z_\lambda)_{\lambda\in[-\pi,\pi]}$ be a Gramian-orthogonal increment process as in Definition 4.6. Then there exists a unique $\mathcal{H}_0$-valued random c.a.g.o.s. W on $((-\pi,\pi],\mathcal{B}((-\pi,\pi]),\Omega,\mathcal{F},\mathbb{P})$ such that (4.11) holds.*

Proof. By (ii) in Definition 4.6, we have that, for all $s<t$ in $[-\pi,\pi]$,

$$\mathbb{E}\left[\|Z_t-Z_{-\pi}\|_{\mathcal{H}_0}^2\right]=\mathbb{E}\left[\|Z_s-Z_{-\pi}\|_{\mathcal{H}_0}^2\right]+\mathbb{E}\left[\|Z_t-Z_s\|_{\mathcal{H}_0}^2\right]\;.$$

Thus, with (iii), we have that the function $F:[-\pi,\pi]\to\mathbb{R}_+$ defined by

$$F(\lambda)=\mathbb{E}\left[\|Z_\lambda-Z_{-\pi}\|_{\mathcal{H}_0}^2\right]$$

is non-decreasing and right-continuous, and it follows that there exists a finite non-negative measure ν on $((-\pi,\pi],\mathcal{B}((-\pi,\pi]))$ such that, for all $s<t$ in $[-\pi,\pi]$,

$$\mathbb{E}\left[\|Z_t-Z_s\|_{\mathcal{H}_0}^2\right]=\nu((s,t])\;.$$

Another straightforward consequence of (ii) in Definition 4.6 is that, for all $s<t$ and $s'<t'$ in $(-\pi,\pi]$, we have

$$\mathbb{E}\left[\langle Z_t-Z_s,Z_{t'}-Z_{s'}\rangle_{\mathcal{H}_0}\right]=\begin{cases}\mathbb{E}\left[\|Z_{t'\wedge t}-Z_{s'\vee s}\|_{\mathcal{H}_0}^2\right] & \text{if } s'\vee s<t'\wedge t,\\ 0 & \text{otherwise.}\end{cases}$$

Thus we can consider the mapping $1_{(s,t]}\mapsto Z(t)-Z(s)$ defined for all $s<t$ in $(-\pi,\pi]$ as a $\mathcal{G}:=L^2((-\pi,\pi],\mathcal{B}((-\pi,\pi]),\nu)\to\mathcal{H}:=\mathcal{M}(\Omega,\mathcal{F},\mathcal{H}_0,\mathbb{P})$ mapping, and, interpreting the right-hand side of the previous display as $\left\langle 1_{(s,t]},1_{(s',t']}\right\rangle_{\mathcal{G}}$, we see that this mapping is isometric. Let us denote by I the unique isometric extension of this mapping on the linear closure of $\left\{1_{(s,t]}\;:\;s<t\in(-\pi,\pi]\right\}$ in $\mathcal{G}$, which happens to be $\mathcal{G}$ itself. We then set, for all $A\in\mathcal{B}((-\pi,\pi])$,

$$W(A)=I(1_A)\;,$$

and we immediately obtain that W is an $\mathcal{H}_0$-valued random c.a.o.s. measure on $((-\pi,\pi],\mathcal{B}((-\pi,\pi]),\Omega,\mathcal{F},\mathbb{P})$ as in Definition 3.1 and its intensity measure is ν. By uniqueness of the isometric extension, it only remains to show that W is moreover a c.a.g.o.s. measure, that is, for all $A,B\in\mathcal{B}((-\pi,\pi])$ such that $A\cap B=\emptyset$, we have

$$[W(A),W(B)]_{\mathcal{H}}=\mathrm{Cov}\left(W(A),W(B)\right)=0\;.$$

This is implied by showing that, for all $x\in\mathcal{H}_0$ such that $\|x\|_{\mathcal{H}_0}=1$, for all $A,B\in\mathcal{B}((-\pi,\pi])$ such that $A\cap B=\emptyset$, we have

$$x^{\mathsf{H}}\mathrm{Cov}\left(W(A),W(B)\right)x=\mathrm{Cov}\left(\langle W(A),x\rangle_{\mathcal{H}_0},\langle W(B),x\rangle_{\mathcal{H}_0}\right)=0\;. \tag{5.7}$$

Now take $x\in\mathcal{H}_0$ such that $\|x\|_{\mathcal{H}_0}=1$ and define $F_x:[-\pi,\pi]\to\mathbb{R}_+$ by

$$F_x(\lambda)=\mathbb{E}\left[\left|\langle Z_\lambda-Z_{-\pi},x\rangle_{\mathcal{H}_0}\right|^2\right]\;.$$

As previously with F, (ii) and (iii) in Definition 4.6 imply that F_x is non-decreasing and right-continuous and it follows that there exists a finite non-negative measure ν_x on $((-\pi,\pi],\mathcal{B}((-\pi,\pi]))$ such that, for all $s<t$ in $[-\pi,\pi]$,

$$\mathbb{E}\left[\left|\langle Z_t - Z_s, x\rangle_{\mathcal{H}_0}\right|^2\right] = \nu_x((s,t]) \ .$$

Again, we can extend the mapping $1_{(s,t]} \mapsto \langle Z_t - Z_s, x\rangle_{\mathcal{H}_0}$ defined for all $s<t$ in $(-\pi,\pi]$ as a $\mathcal{G}_x := L^2((-\pi,\pi],\mathcal{B}((-\pi,\pi]),\nu_x) \to \mathcal{M}(\Omega,\mathcal{F},\mathbb{C},\mathbb{P})$ isometric mapping, which we denote by I_x in the following. We further denote by W_x the c.a.o.s. measure defined by $W_x(A) = I_x(1_A)$ for all $A \in \mathcal{B}((-\pi,\pi])$. This is a $\mathbb{C}$-valued random c.a.o.s. measure on $((-\pi,\pi],\mathcal{B}((-\pi,\pi]),\Omega,\mathcal{F},\mathbb{P})$ with intensity measure ν_x. Hence to obtain (5.7) and conclude the proof, we only need to show that for all $A \in \mathcal{B}((-\pi,\pi])$, we have

$$W_x(A) = \langle W(A), x\rangle_{\mathcal{H}_0} \ . \tag{5.8}$$

We already know that this is true for $A \in \mathcal{C} = \{(-\pi,\lambda] \,:\, \lambda \in (-\pi,\pi]\}$ by definitions of W_x, W, I_x and I. The class $\mathcal{C}$ is a π-system of Borel sets and satisfies $\sigma(\mathcal{C}) = \mathcal{B}((-\pi,\pi])$. We conclude with the π-λ-theorem by observing that the class $\mathcal{A}$ of sets $A \in \mathcal{B}((-\pi,\pi])$, such that (5.8) holds is a λ-system. Indeed if $A \in \mathcal{A}$, then $A^c = (-\pi,\pi] \setminus A$ satisfies $W_x(A^c) = W_x((-\pi,\pi]) - W_x(A)$ and $W(A^c) = W((-\pi,\pi]) - W(A)$, so that $A, (-\pi,\pi] \in \mathcal{A}$ implies $A^c \in \mathcal{A}$. Similarly, if $(A_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ with $A_n \cap A_p = \emptyset$ for $n \neq p$ then $\cup_n A_n \in \mathcal{A}$ because the c.a.o.s. measures W_x and W are σ-additive in $\mathcal{M}(\Omega,\mathcal{F},\mathbb{C},\mathbb{P})$ and in $\mathcal{M}(\Omega,\mathcal{F},\mathcal{H}_0,\mathbb{P})$, respectively, see Remark 3.1. ☐

5.3 Proofs of Section 4.1

Let us start with the proof of the Gramian-Cramér representation theorem, as a consequence of the Stone theorem. The usual Stone theorem (see e.g. [6, Chapter IX]) says that any continuous isomorphism $h \mapsto U_h$ from an l.c.a. group G to the set of unitary operators from a Hilbert space $\mathcal{H}$ onto itself can be represented as an integral of this mapping, that is,

$$U_h = \int \chi(h)\, \xi(\mathrm{d}\chi) \ ,$$

where ξ is a p.o.v.m. defined on the dual set of characters $\hat{\mathrm{G}}$ endowed with its Borel σ-field and valued in the set of orthogonal projections on $\mathcal{H}$. This classical theorem has a counterpart in the case where $\mathcal{H}$ is an $\mathcal{L}_b(\mathcal{H}_0)$-normal Hilbert module and each U_h is not only unitary but also Gramian-unitary, in which case ξ is valued in the set of orthogonal projections on $\mathcal{H}$ whose ranges are closed submodules. See [12, Section 2.5] for details. It turns out that such p.o.v.m.'s are related to c.a.g.o.s. measure by the following lemma.

Lemma 5.4. *Let $\mathcal{H}_0$ be a separable Hilbert space, $\mathcal{H}$ an $\mathcal{L}_b(\mathcal{H}_0)$-normal Hilbert module and $(\Lambda,\mathcal{A})$ a measurable space. Let ξ be a p.o.v.m. on $(\Lambda,\mathcal{A},\mathcal{H})$ valued in the set of orthogonal projections on $\mathcal{H}$ whose ranges are closed submodules. Then for all $x_0 \in \mathcal{H}$, the mapping $\xi x_0 : A \mapsto \xi(A)x_0$ is a c.a.g.o.s. measure on $(\Lambda,\mathcal{A},\mathcal{H})$ which is regular if ξ is regular.*

Proof. Using the fact that ξ is a p.o.v.m. on $(\Lambda,\mathcal{A},\mathcal{H})$ and [3, Proposition 1], it is straightforward to see that ξx_0 is an $\mathcal{H}$-valued measure. Moreover, since ξ is valued in the set of orthogonal projection on $\mathcal{H}$ whose ranges are closed submodules, we get that for all disjoint $A, B \in \mathcal{B}(\mathrm{G})$

$$[\xi(A)x_0, \xi(B)x_0]_{\mathcal{H}} = [\xi(B)\xi(A)x_0, x_0]_{\mathcal{H}} = [\xi(B\cap A)x_0, x_0]_{\mathcal{H}} = 0 \ ,$$

where the first equality is justified in [12, P. 23] and the second one by [3, Theorem 3]. This proves that ξx_0 is a c.a.g.o.s. measure on $(\Lambda,\mathcal{A},\mathcal{H})$. In the following, we denote by ν its intensity operator measure. Then, for all $A \in \mathcal{A}$, we have

$$\|\nu(A)\|_1 = \mathrm{Tr}[\xi(A)x_0, \xi(A)x_0]_{\mathcal{H}} = \langle \xi(A)x_0, x_0\rangle_{\mathcal{H}} \ ,$$

where the last equality comes from the fact that $\xi(A)$ is an orthogonal projection on $\mathcal{H}$. Now, if ξ is regular, then the measure $A \mapsto \langle \xi(A)x_0, x_0\rangle$ is regular and so is $\|\nu\|_1$ by the previous display. This implies that ξx_0 is regular and the proof is concluded. ☐

Proof of Theorem 4.1. Suppose that X is weakly stationary as in Definition 4.1. Then the collection of lag operators $(U_h^X)_{h\in\mathrm{G}}$ of Remark 4.2 satisfies the assumptions of the generalized Stone's theorem stated as Proposition 4 in [12, Section 2.5]. This gives that there exists a regular p.o.v.m. ξ^X on $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{H}^X)$ valued in the set of orthogonal projections whose ranges are closed submodules of $\mathcal{H}^X$ such that, for all $h \in \mathrm{G}$,

$$U_h^X = \int \chi(h)\, \xi^X(\mathrm{d}\chi) \,, \tag{5.9}$$

where the integral is as in Definition 2.5. Then, by Lemma 5.4, the mapping

$$\hat{X} : \begin{array}{ccc} \mathcal{B}(\hat{\mathrm{G}}) & \to & \mathcal{H}^X \\ A & \mapsto & \xi^X(A)X_0 \end{array} \tag{5.10}$$

is a regular c.a.g.o.s. measure on $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{H}^X)$ and we denote by ν_X its intensity operator measure. Since $\mathcal{H}^X$ is a submodule of $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$, $\hat{X}$ is also a regular $\mathcal{H}_0$-valued random c.a.g.o.s. measure on $(\Omega, \mathcal{F}, \mathbb{P})$, see Definition 3.2. Relation (4.2) then follows by applying (5.9) and the fact that, for all $t \in \mathrm{G}$, $U_t^h X_0 = X_t$ and, for all $\phi : \Lambda \to \mathbb{C}$ measurable and bounded,

$$\int \phi \,\mathrm{d}\hat{X} = \left(\int \phi \,\mathrm{d}\xi^X\right) X_0 \,, \tag{5.11}$$

where the integral in the left-hand side is defined as in Definition 3.4 (see also Remark 3.3) and the integral in the right-hand side as in Definition 2.5, for the p.o.v.m. ξ^X. Relation (5.11) obviously holds if $\phi = \mathbb{1}_A$ with $A \in \mathcal{A}$ and also for ϕ simple by linearity. Now, for a general measurable and bounded $\phi : \Lambda \to \mathbb{C}$, we can find a sequence $(\phi_n)_{n\in\mathbb{N}}$ of simple functions such that $|\phi_n| \leq |\phi|$ for all $n \in \mathbb{N}$ and $\phi_n(\lambda) \to \phi(\lambda)$ as $n \to \infty$ for all $\lambda \in \Lambda$. Then, by dominated convergence, ϕ_n converges to ϕ in $L^2(\Lambda, \mathcal{A}, \|\nu\|_1)$ and therefore $\phi_n \mathrm{Id}$ converges to $\phi\mathrm{Id}$ in $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0), \nu)$. Thus $\int \phi_n \,\mathrm{d}\hat{X} \to \int \phi \,\mathrm{d}\hat{X}$ in $\mathcal{H}^X$ by the isometric property of the integral of Definition 3.4. To get (5.11), it now suffices to show that, for all $Y \in \mathcal{H}^X$, $\left\langle \left(\int \phi_n \,\mathrm{d}\xi\right) X_0, Y\right\rangle_{\mathcal{H}^X} \to \left\langle \left(\int \phi \,\mathrm{d}\xi\right) X_0, Y\right\rangle_{\mathcal{H}^X}$. This follows from the polarization formula, Definition 2.5 and dominated convergence.

To show uniqueness, suppose there exists another regular $\mathcal{H}_0$-valued random c.a.g.o.s. measure W on $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \Omega, \mathcal{F}, \mathbb{P})$ satisfying the same identity as (4.2) with $\hat{X}$ replaced by W. Then, we get

$$\int \chi(t)\; \hat{X}(\mathrm{d}\chi) = \int \chi(t)\; W(\mathrm{d}\chi) \quad \text{for all} \quad t \in \mathrm{G} \,. \tag{5.12}$$

Let η denote the Haar measure on G and denote by $\mathcal{C}_c(\mathrm{G})$ the space of compactly supported functions from G to $\mathbb{C}$. Then, by [24, Theorem 1.2.4] and [24, Section E.8], the space

$$E = \left\{\hat{\phi} : \chi \mapsto \int \phi(t)\overline{\chi(t)}\; \eta(\mathrm{d}t) \;:\; \phi \in L^1(\mathrm{G}, \mathcal{B}(\mathrm{G}), \eta)\right\}$$

is dense in $L^2(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \|\nu_W\|_1 + \|\nu_X\|_1)$. We can thus find, for any $A \in \mathcal{B}(\hat{\mathrm{G}})$, $(\phi_n)_{n\in\mathbb{N}} \in \mathcal{C}_c(\mathrm{G})^{\mathbb{N}}$ such that, defining $\hat{\phi}_n$ as above, $\hat{\phi}_n \to \mathbb{1}_A$ both in $L^2(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \|\nu_W\|_1)$ and in $L^2(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \|\nu_X\|_1)$. Then by Proposition 3.7, we have, for all $n \in \mathbb{N}$,

$$\begin{aligned}\int \hat{\phi}_n(\chi)\; W(\mathrm{d}\chi) &= \int \left(\int \chi(-t)\; W(\mathrm{d}\chi)\right) \phi_n(t)\; \eta(\mathrm{d}t) \\ &= \int \left(\int \chi(-t)\; \hat{X}(\mathrm{d}\chi)\right) \phi_n(t)\; \eta(\mathrm{d}t) = \int \hat{\phi}_n(\chi)\; \hat{X}(\mathrm{d}\chi) \,,\end{aligned}$$

where we have used (5.12) in the second equality. Letting $n \to \infty$, we get $W(A) = \hat{X}(A)$, thus proving the uniqueness of $\hat{X}$. □

We can now prove the Kolmogorov isomorphism theorem.

Proof of Theorem 4.3. By Theorem 3.5 and (4.4), $I_{\hat{X}}^{\mathcal{G}_0}$ is a Gramian-unitary operator from $\widehat{\mathcal{H}}^{X,\mathcal{G}_0}$ to $\mathcal{H}^{\hat{X},\mathcal{G}_0}$. Thus to conclude, we only need to show that $\mathcal{H}^{X,\mathcal{G}_0} = \mathcal{H}^{\hat{X},\mathcal{G}_0}$. By (4.2), we have for all $\mathrm{P} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and $t \in \mathrm{G}$, $\mathrm{P}X_t = I_{\hat{X}}^{\mathcal{G}_0}(\mathrm{Pe}_t) \in \mathcal{H}^{\hat{X},\mathcal{G}_0}$, where $\mathrm{e}_t : \chi \mapsto \chi(t)$. Thus, by (4.1), we get that $\mathcal{H}^{X,\mathcal{G}_0} \subset \mathcal{H}^{\hat{X},\mathcal{G}_0}$. The definition of $\hat{X}$ in (5.10) gives the converse inclusion, which achieves the proof. □

We already provided a proof of Theorem 4.6, mainly based on [4]. We hereafter propose an alternative and more elementary proof of one of the implications in Theorem 4.6.

Proof of (iv)⇒(ii) in Theorem 4.6. Suppose that (iv) holds. The continuity of Γ in w.o.t. follows immediately by dominated convergence and we now prove that it is of positive type as in Definition 4.5. Take some arbitrary $n \in \mathbb{N}^*$, and $x_1, \cdots, x_n \in \mathcal{H}_0$. Let us define the $\mathbb{C}^{n\times n}$-valued measure μ on on $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}))$ by

$$\mu(A) = \begin{bmatrix} \langle \nu(A)x_1, x_1 \rangle_{\mathcal{H}_0} & \cdots & \langle \nu(A)x_n, x_1 \rangle_{\mathcal{H}_0} \\ \vdots & \ddots & \vdots \\ \langle \nu(A)x_1, x_n \rangle_{\mathcal{H}_0} & \cdots & \langle \nu(A)x_n, x_n \rangle_{\mathcal{H}_0} \end{bmatrix} .$$

Then, by the Cauchy-Schwartz inequality, for all $i, j \in [\![1, n]\!]$, the $\mathbb{C}$-valued measure $\mu_{i,j}$: $A \mapsto [\mu(A)]_{i,j}$ admits a density $f_{i,j}$ with respect to the non-negative finite measure $\|\mu\|_1$: $A \mapsto \|\mu(A)\|_1 = \mathrm{Tr}(\mu(A))$ and the matrix-valued function $f : \chi \mapsto (f_{i,j}(\chi))_{1\leq i,j\leq n}$ is $\|\mu\|_1$-a.e. hermitian, non-negative semi-definite since, for all $a \in \mathbb{C}^n$ and $A \in \mathcal{B}(\hat{\mathrm{G}})$,

$$\int_A a^{\mathsf{H}} f(\chi) a \; \|\mu\|_1(\mathrm{d}\chi) = a^{\mathsf{H}} \mu(A) a = \left(\sum_{i=1}^n a_i x_i \right)^{\mathsf{H}} \nu(A) \left(\sum_{i=1}^n a_i x_i \right) \geq 0 .$$

Then, for all $t_1, \cdots, t_n \in \mathrm{G}$, we have

$$\begin{aligned} \sum_{i,j=1}^n \langle \Gamma(t_i - t_j) x_i, x_j \rangle_{\mathcal{H}_0} &= \sum_{i,j=1}^n \int \chi(t_i) \overline{\chi(t_j)} \, \mu_{i,j}(\mathrm{d}\chi) \\ &= \sum_{i,j=1}^n \int \chi(t_i) \overline{\chi(t_j)} f_{i,j}(\chi) \, \|\mu\|_1(\mathrm{d}\chi) \\ &= \int \underbrace{\left(\sum_{i,j=1}^n \chi(t_i) \overline{\chi(t_j)} f_{i,j}(\chi) \right)}_{\geq 0 \; \|\mu\|_1\text{-a.e.}} \|\mu\|_1(\mathrm{d}\chi) \\ &\geq 0 . \end{aligned}$$

The first line follows from (iv), the definition of $\mu_{i,j}$ above and the definition of the integral as given by Definition 2.5. The second line follows from the definition of $f_{i,j}$ and the third line from the above property of the matrix-valued function f. Hence we have shown (ii) and the proof of the implication is concluded. □

5.4 Composition and inversion of filters of random c.a.g.o.s. measures and proofs of Section 4.2

Having a clear description of the modular spectral domain at hand, the results of Section 4.2, mainly Proposition 4.8, can be seen as a particular instance of the composition and inversion of operator valued functions filtering a general random c.a.g.o.s. measure, which is the framework of this section. Namely, consider the filtering (using Definition 3.5)

$$V = \hat{F}_{\Phi}(W)$$

for a random c.a.g.o.s. measure W and a transfer function $\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$. The goal of this section is, given another separable Hilbert space $\mathcal{I}_0$, to characterize the transfer functions Ψ valued in $\mathcal{O}(\mathcal{G}_0, \mathcal{I}_0)$ which can be used to filter the c.a.g.o.s. measure V. Taking W to be the Cramér representation $\hat{X}$ of a weakly stationary process X, we will get the already stated Proposition 4.8 on the composition of linear filters as a by-product.

According to Proposition 3.6, Ψ must be square-integrable with respect to $\nu_V = \Phi \nu_W \Phi^{\mathsf{H}}$ and this turns out to be equivalent to checking that $\Psi\Phi$ is square integrable with respect to ν_W as stated in the following theorem. We recall that $\Psi\Phi$ is the pointwise composition, that is, $\Psi\Phi : \lambda \mapsto \Psi(\lambda) \circ \Phi(\lambda)$ and is defined whenever the image of $\Phi(\lambda)$ is included in the domain of $\Psi(\lambda)$.

We first need the following lemma, which will be used in the proof of Theorem 5.6.

Lemma 5.5. *Let $\mathcal{H}_0$, $\mathcal{G}_0$, $\mathcal{I}_0$ be separable Hilbert spaces and* $\mathrm{P} \in \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0)$*,* $\mathrm{Q} \in \mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$*. The following assertions hold.*

(i) $\mathrm{Im}(|\mathrm{Q}^{\mathsf{H}}|) = \mathrm{Im}(\mathrm{Q})$.

(ii) If $\mathrm{Im}(\mathrm{Q}) \subset \mathcal{D}(\mathrm{P})$*, then* $(\mathrm{PQ})(\mathrm{PQ})^{\mathsf{H}} = (\mathrm{P}\,|\mathrm{Q}^{\mathsf{H}}|)(\mathrm{P}\,|\mathrm{Q}^{\mathsf{H}}|)^{\mathsf{H}}$.

(iii) If $\mathrm{Im}(\mathrm{Q}) \subset \mathcal{D}(\mathrm{P})$*, then* $\mathrm{PQ} \in \mathcal{S}_2(\mathcal{H}_0, \mathcal{I}_0)$ *if and only if* $\mathrm{P}\,|\mathrm{Q}^{\mathsf{H}}| \in \mathcal{S}_2(\mathcal{G}_0, \mathcal{I}_0)$*. In this case* $\|\mathrm{PQ}\|_2 = \|\mathrm{P}\,|\mathrm{Q}^{\mathsf{H}}|\|_2$.

Proof. For convenience, we only consider the case where the spaces have infinite dimensions. The singular values decomposition of Q yields for two orthonormal sequences $(\psi_n)_{n\in\mathbb{N}} \in \mathcal{G}_0^{\mathbb{N}}$ and $(\phi_n)_{n\in\mathbb{N}} \in \mathcal{H}_0^{\mathbb{N}}$,

$$\mathrm{Q} = \sum_{n\in\mathbb{N}} \sigma_n \psi_n \otimes \phi_n \quad \text{and} \quad \left|\mathrm{Q}^{\mathsf{H}}\right| = \sum_{n\in\mathbb{N}} \sigma_n \psi_n \otimes \psi_n \;.$$

Proof of (i). We have $\mathrm{Im}(\mathrm{Q}) = \left\{\sum_{n\in\mathbb{N}} \sigma_n x_n \psi_n \;:\; (x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})\right\} = \mathrm{Im}(|\mathrm{Q}^{\mathsf{H}}|)$.
Proof of (ii). By the first point both compositions PQ and $\mathrm{P}\,|\mathrm{Q}^{\mathsf{H}}|$ make sense. Consider the polar decomposition of Q^{H} : $\mathrm{Q}^{\mathsf{H}} = U\,|\mathrm{Q}^{\mathsf{H}}|$, with $U = \sum_{n\in\mathbb{N}} \phi_n \otimes \psi_n$. Then $\mathrm{Q} = |\mathrm{Q}^{\mathsf{H}}|\, U^{\mathsf{H}}$ and

$$(\mathrm{PQ})(\mathrm{PQ})^{\mathsf{H}} = \left(\mathrm{P}\left|\mathrm{Q}^{\mathsf{H}}\right|\right) U^{\mathsf{H}} U \left(\mathrm{P}\left|\mathrm{Q}^{\mathsf{H}}\right|\right)^{\mathsf{H}} = \left(\mathrm{P}\left|\mathrm{Q}^{\mathsf{H}}\right|\right)\left(\mathrm{P}\left|\mathrm{Q}^{\mathsf{H}}\right|\right)^{\mathsf{H}} \;,$$

where we used that $|\mathrm{Q}^{\mathsf{H}}|\, U^{\mathsf{H}} U = |\mathrm{Q}^{\mathsf{H}}|$.
Proof of (iii). We have that $\mathrm{PQ} \in \mathcal{S}_2(\mathcal{H}_0, \mathcal{I}_0)$ if and only if $(\mathrm{PQ})(\mathrm{PQ})^{\mathsf{H}} \in \mathcal{S}_1(\mathcal{I}_0)$, which is equivalent to $\mathrm{P}\,|\mathrm{Q}^{\mathsf{H}}| \in \mathcal{S}_2(\mathcal{G}_0, \mathcal{I}_0)$ by the previous point. □

We can now derive the main result of this section.

Theorem 5.6. *Let $(\Lambda, \mathcal{A})$ be a measurable space, $\mathcal{H}_0$, $\mathcal{G}_0$, $\mathcal{I}_0$ separable Hilbert spaces and ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. Let $\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ and $\Psi \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{G}_0, \mathcal{I}_0)$. Define $\Phi\nu\Phi^{\mathsf{H}} : A \mapsto \int_A \Phi \mathrm{d}\nu \Phi^{\mathsf{H}} = [\mathbb{1}_A \Phi, \mathbb{1}_A \Phi]_{\nu}$, which is a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{G}_0)$. Then*

$$\Psi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}}) \Leftrightarrow \Psi\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \nu) \;. \tag{5.13}$$

Moreover, the following assertions hold.

(a) For all $\Psi, \Theta \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}})$*,*

$$(\Psi\Phi)\nu(\Theta\Phi)^{\mathsf{H}} = \Psi(\Phi\nu\Phi^{\mathsf{H}})\Theta^{\mathsf{H}} \;.$$

(b) The mapping $\Psi \mapsto \Psi\Phi$ *is a well defined Gramian-isometric operator from* $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}})$ *to* $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \nu)$.

(c) Suppose moreover that Φ is injective $\|\nu\|_1$-a.e., then we have that

$$\Phi^{-1} \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{H}_0), \Phi\nu\Phi^{\mathsf{H}}) \;,$$

where we define $\Phi^{-1}(\lambda) := \left(\Phi(\lambda)_{|\mathcal{D}(\Phi(\lambda))\to\mathrm{Im}(\Phi(\lambda))}\right)^{-1}$ *with domain* $\mathrm{Im}(\Phi(\lambda))$ *for all* $\lambda \in \{\Phi \text{ is injective}\}$ *and* $\Phi^{-1}(\lambda) = 0$ *otherwise.*

Proof. Let μ be a dominating measure for $\|\nu\|_1$ and $g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$, then, by definition of $\Phi\nu\Phi^{\mathsf{H}}$, μ also dominates $\|\Phi\nu\Phi^{\mathsf{H}}\|_1$ and $\frac{\mathrm{d}\Phi\nu\Phi^{\mathsf{H}}}{\mathrm{d}\mu} = (\Phi g^{1/2})(\Phi g^{1/2})^{\mathsf{H}}$. Hence, $\left(\frac{\mathrm{d}\Phi\nu\Phi^{\mathsf{H}}}{\mathrm{d}\mu}\right)^{1/2} = \left|(\Phi g^{1/2})^{\mathsf{H}}\right|$ and we get, by Proposition 3.4,

$$\begin{aligned}
\Psi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}}) &\Leftrightarrow \begin{cases} \mathrm{Im}\left|(\Phi g^{1/2})^{\mathsf{H}}\right| \subset \mathcal{D}(\Psi) \;\; \mu\text{-a.e.} \\ \Psi\left|(\Phi g^{1/2})^{\mathsf{H}}\right| \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{G}_0, \mathcal{I}_0), \mu) \end{cases} \\
&\Leftrightarrow \begin{cases} \mathrm{Im} g^{1/2} \subset \mathcal{D}(\Psi\Phi) \;\; \mu\text{-a.e.} \\ \Psi\Phi g^{1/2} \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{I}_0), \mu) \end{cases} \\
&\Leftrightarrow \Psi\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \nu) \;,
\end{aligned}$$

where the second equivalence comes from Lemma 5.5 and the fact that for all $\lambda \in \Lambda$, $\mathcal{D}(\Psi(\lambda)\Phi(\lambda))$ is the preimage of $\mathcal{D}(\Psi(\lambda))$ by $\Phi(\lambda)$ which gives that $\mathrm{Im}(g^{1/2}(\lambda)) \subset \mathcal{D}(\Psi(\lambda)\Phi(\lambda))$ if and only if $\mathrm{Im}(\Phi(\lambda)g^{1/2}(\lambda)) \subset \mathcal{D}(\Psi(\lambda))$.

Let us now prove Assertion (a). For all $\Psi, \Theta \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}})$ and $A \in \mathcal{A}$,

$$
\begin{aligned}
(\Psi\Phi)\nu(\Theta\Phi)^{\mathsf{H}}(A) &= \int_A \left(\Psi\Phi g^{1/2}\right)\left(\Theta\Phi g^{1/2}\right)^{\mathsf{H}} \mathrm{d}\mu \\
&= \int_A \left(\Psi \left|(\Phi g^{1/2})^{\mathsf{H}}\right|\right)\left(\Theta \left|(\Phi g^{1/2})^{\mathsf{H}}\right|\right)^{\mathsf{H}} \mathrm{d}\mu \qquad \text{(by lemma 5.5)} \\
&= \Psi(\Phi\nu\Phi^{\mathsf{H}})\Theta^{\mathsf{H}}(A)\;.
\end{aligned}
$$

Assertion (a) follows as well as Assertion (b) by taking $A = \Lambda$. Finally, to show Assertion (c), suppose that Φ is injective $\|\nu\|_1$-a.e. then $\Phi^{-1}\Phi : \lambda \mapsto \mathrm{Id}_{\mathcal{H}_0}\mathbb{1}_{\{\Phi(\lambda) \text{ is injective}\}}$ is in $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0), \nu)$ which gives that $\Phi^{-1} \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{H}_0), \Phi\nu\Phi^{\mathsf{H}})$ by Assertion (a). □

We deduce the following corollary on the composition and inversion for random c.a.g.o.s. measures.

Corollary 5.7 (Composition and inversion of filters on random c.a.g.o.s. measures)**.** *Let $(\Lambda, \mathcal{A})$ be a measurable space, $\mathcal{H}_0$, $\mathcal{G}_0$ two separable Hilbert spaces, and $\Phi \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$. Let $W \in \hat{\mathcal{S}}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure ν_W. Then three following assertions hold.*

(i) For any separable Hilbert space $\mathcal{I}_0$, we have $\mathcal{H}^{\hat{F}_\Phi(W), \mathcal{I}_0} \underset{\approx}{\subseteq} \mathcal{H}^{W, \mathcal{I}_0}$.

(ii) For any separable Hilbert space $\mathcal{I}_0$ and all $\Psi \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{G}_0, \mathcal{I}_0)$, we have $W \in \hat{\mathcal{S}}_{\Psi\Phi}(\Omega, \mathcal{F}, \mathbb{P})$ if and only if $\hat{F}_\Phi(W) \in \hat{\mathcal{S}}_\Psi(\Omega, \mathcal{F}, \mathbb{P})$, and in this case, we have

$$\hat{F}_\Psi \circ \hat{F}_\Phi(W) = \hat{F}_{\Psi\Phi}(W). \tag{5.14}$$

(iii) Suppose that Φ is injective $\|\nu_W\|_1$-a.e. Then $W = F_{\Phi^{-1}} \circ F_\Phi(W)$, where Φ^{-1} is defined as in Assertion (c) of Theorem 5.6. Moreover, Assertion (i) above holds with $\underset{\approx}{\subseteq}$ replaced by $\cong$.

Proof. **Proof of Assertion (i).** This follows from Assertion (b) of Theorem 5.6 and Theorem 3.5.

Proof of Assertion (ii). If $W \in \hat{\mathcal{S}}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$, then the equivalence between $W \in \hat{\mathcal{S}}_{\Psi\Phi}(\Omega, \mathcal{F}, \mathbb{P})$ and $\hat{F}_\Phi(W) \in \hat{\mathcal{S}}_\Psi(\Omega, \mathcal{F}, \mathbb{P})$ is just another formulation of the equivalence (5.13) with $\nu = \nu_W$. Suppose that it holds and set $V := \hat{F}_\Phi(W)$ so that $\nu_V = \Phi\nu\Phi^{\mathsf{H}}$ and (5.14) means that, for all $\Psi \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \nu_V)$ and $A \in \mathcal{A}$, $\int_A \Psi \,\mathrm{d}V = \int_A \Psi\Phi \,\mathrm{d}W$. Replacing Ψ by $\Psi\mathbb{1}_A$, it is sufficient to show this identity with $A = \Lambda$. Using that the integral with respect to a random c.a.g.o.s. measure is Gramian-isometric and Assertion (b) of Theorem 5.6, the mappings $\Psi \mapsto \int \Psi \,\mathrm{d}V$ and $\Psi \mapsto \int \Psi\Phi \,\mathrm{d}W$ are Gramian-isometric from $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu_W\Phi^{\mathsf{H}})$ to $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{I}_0, \mathbb{P})$. Hence by Theorem 3.3, they coincide on the whole space if they coincide on all $\Psi = \mathbb{1}_A \mathrm{P}$ for $A \in \mathcal{A}$ and $\mathrm{P} \in \mathcal{L}_b(\mathcal{G}_0, \mathcal{I}_0)$. To conclude the proof of Assertion (ii), it is thus enough to prove that, for all $A \in \mathcal{A}$ and $\mathrm{P} \in \mathcal{L}_b(\mathcal{G}_0, \mathcal{I}_0)$,

$$\int_A \mathrm{P}\,\mathrm{d}V = \int_A \mathrm{P}\Phi\,\mathrm{d}W\;.$$

This identity follows from the definition of V and the fact that on both sides the operator P can be moved in front of the integrals. This latter fact directly follows from the definition of the integral for the left-hand side and for the right-hand side when $\Phi = \mathbb{1}_B$ for some $B \in \mathcal{A}$, which extends to all Φ by observing that $\Phi \mapsto \int \mathrm{P}\Phi \,\mathrm{d}W$ and $\Phi \mapsto \mathrm{P}\int \Phi \,\mathrm{d}W$ are continuous on $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$.

Proof of Assertion (iii). Continuing with the setting of the proof of the previous point, we now suppose that Φ is injective $\|\nu_W\|_1$-a.e. Assertions (c) and (a) of Theorem 5.6 give that $\Phi^{-1} \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{H}_0), \nu_V)$ (*i.e.* $V \in \hat{\mathcal{S}}_{\Phi^{-1}}(\Omega, \mathcal{F}, \mathbb{P})$) and $\Phi^{-1}\nu_V\left(\Phi^{-1}\right)^{\mathsf{H}} = \nu_W$. Hence, writing Relation (5.14) with $\Psi = \Phi^{-1}$, we get $\hat{F}_{\Phi^{-1}}(V) = \hat{F}_{\Phi^{-1}\Phi}(W) = W$. Moreover, reversing the roles of W and V in assertion (i) gives the embedding $\mathcal{H}^{W, \mathcal{I}_0} \underset{\approx}{\subseteq} \mathcal{H}^{\hat{F}_\Phi(W), \mathcal{I}_0}$ which, with Assertion (i), allow us to conclude that $\mathcal{H}^{W, \mathcal{I}_0} \cong \mathcal{H}^{\hat{F}_\Phi(W), \mathcal{I}_0}$. □

We conclude this section with the proof of Proposition 4.8.

Proof of Proposition 4.8. Using the Gramian-unitary operator between the modular time domain and the modular spectral domain, this result is a direct application of Corollary 5.7 with $\Lambda = \hat{\mathrm{G}}$ and $\mathcal{A} = \mathcal{B}(\hat{\mathrm{G}})$ and $W = \hat{X}$. $\square$

5.5 Proofs of Section 4.3

The goal of this section is to provide a proof of Lemma 4.9. Before that, let us recall essential facts about the diagonalization of compact positive operators. Let $\mathcal{H}_0$ be a separable Hilbert space of dimension $N \in \{1, \cdots, +\infty\}$, $(\Lambda, \mathcal{A})$ be a measurable space and $\Phi \in \mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}_0)$ such that for all $\lambda \in \Lambda$, $\Phi(\lambda) \in \mathcal{S}_1^+(\mathcal{H}_0)$. Then, in this case, for any $\lambda \in \Lambda$, $\Phi(\lambda)$ admits the eigendecomposition

$$\Phi(\lambda) = \sum_{0 \le n < N} \sigma_n(\lambda) \phi_n(\lambda) \otimes \phi_n(\lambda) \;, \tag{5.15}$$

where the series converges in operator norm and the family $(\phi_n(\lambda))_{0 \le n < N}$ is orthonormal. Moreover, we have

$$\mathrm{Tr}(\Phi(\lambda)) = \sum_{0 \le n < N} \sigma_n(\lambda) < +\infty \;.$$

The following theorem shows that such a decomposition can be constructed in a way which makes the eigenvalues and eigenvectors measurable as functions of λ.

We will need the following lemmas, which rely on the weak topology on $\mathcal{H}_0$, defined as the smallest topology which makes the functions $\left\{x^{\mathsf{H}} \,:\, x \in \mathcal{H}_0\right\}$ continuous.

Lemma 5.8. *Let $\mathcal{H}_0$ be a separable Hilbert space and denote the closed unit ball by*

$$\bar{B}_{0,1} := \left\{x \in \mathcal{H}_0 \,:\, \|x\|_{\mathcal{H}_0} \le 1\right\} \;.$$

Then $\bar{B}_{0,1}$ endowed with the weak topology is a compact metrizable space.

Proof. By the Banach-Alaoglu theorem, $\bar{B}_{0,1}$ is compact for the weak topology. Since $\mathcal{H}_0$ is separable, we can choose a Hilbert basis $(\psi_n)_{0 \le n < N}$ for $\mathcal{H}_0$, with $N \in \{1, \cdots, +\infty\}$. It is straightforward to show that the mapping $(x, y) \mapsto \sum_{0 \le n < N} 2^{-n} \left|\langle x - y, \psi_n \rangle_{\mathcal{H}_0}\right|$ is a metric inducing the weak topology on $\bar{B}_{0,1}$. $\square$

Lemma 5.9. *Let $\mathcal{H}_0$ be a separable Hilbert space. Then the Borel σ-field $\mathcal{B}_w(\mathcal{H}_0)$ of $\mathcal{H}_0$ endowed with the weak topology coincides with the (usual) Borel σ-field $\mathcal{B}(\mathcal{H}_0)$ of $(\mathcal{H}_0, \|\cdot\|_{\mathcal{H}_0})$.*

Proof. The weak topology is included in the topology of $(\mathcal{H}_0, \|\cdot\|_{\mathcal{H}_0})$, hence $\mathcal{B}_w(\mathcal{H}_0) \subset \mathcal{B}(\mathcal{H}_0)$. To prove the converse inclusion, observe that by expressing $\|x - y\|_{\mathcal{H}_0}$ as the ℓ^2-norm of the inner-products of $(x - y)$ with a Hilbert basis $(\psi_n)_{0 \le n < N}$, we easily get that $y \mapsto \|x - y\|_{\mathcal{H}_0}$ is measurable from $(\mathcal{H}_0, \mathcal{B}_w(\mathcal{H}_0))$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ for all $x \in \mathcal{H}_0$. Hence $\mathcal{B}(\mathcal{H}_0) \subset \mathcal{B}_w(\mathcal{H}_0)$, which concludes the proof. $\square$

Lemma 5.10. *Let $\mathcal{H}_0$ be a separable Hilbert space. If $\mathrm{P} \in \mathcal{S}_1^+(\mathcal{H}_0)$ then the mapping $x \mapsto \langle \mathrm{P}x, x \rangle_{\mathcal{H}_0}$ is continuous on the unit closed ball $\bar{B}_{0,1}$ for the weak topology.*

Proof. Let us consider the eigendecomposition $\mathrm{P} = \sum_{0 \le n < N} \sigma_n \phi_n \otimes \phi_n$. Then for all $x \in \bar{B}_{0,1}$, $\langle \mathrm{P}x, x \rangle_{\mathcal{H}_0} = \sum_{0 \le n < N} \sigma_n \left|\langle x, \phi_n \rangle_{\mathcal{H}_0}\right|^2$ and the result follows by dominated convergence since

$$\sup_{x \in \bar{B}_{0,1}} \left|\langle x, \phi_n \rangle_{\mathcal{H}_0}\right|^2 \le 1 \quad \text{and} \quad \sum_{0 \le n < N} \sigma_n < +\infty \;.$$ $\square$

We can now prove the following theorem.

Theorem 5.11. *Let $\mathcal{H}_0$ be a separable Hilbert space and $(\Lambda, \mathcal{A})$ be a measurable space. Let $\Phi \in \mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}_0)$ such that for all $\lambda \in \Lambda$, $\Phi(\lambda) \in \mathcal{S}_1^+(\mathcal{H}_0)$. Then the pairs $\{(\sigma_n, \phi_n) \,:\, 0 \le n < N\}$ in* (5.15) *can be taken so that for all $0 \le n < N$, σ_n is measurable from $(\Lambda, \mathcal{A})$ to $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ and ϕ_n is measurable from $(\Lambda, \mathcal{A})$ to $(\mathcal{H}_0, \mathcal{B}(\mathcal{H}_0))$.*

Proof. The construction of the eigenvalues and eigenvectors is done iteratively using the Measurable Maximum Theorem [1, Theorem 18.19] on $\Lambda \times \bar{B}_{0,1}$, where $\bar{B}_{0,1}$ denotes the closed unit ball of $\mathcal{H}_0$, which is compact metrizable for the weak topology by Lemma 5.8. As in [1, Definition 17.1], a *correspondence* φ from Λ to $\bar{B}_{0,1}$, denoted by $\varphi : \Lambda \twoheadrightarrow \bar{B}_{0,1}$, is a mapping which assigns each element of Λ to a subset of $\bar{B}_{0,1}$.

Construction of (σ_1, ϕ_1) **:** Define

$$f: \begin{array}{lcl} \Lambda \times \bar{B}_{0,1} & \to & \mathbb{R}_+ \\ (\lambda, x) & \mapsto & \langle \Phi(\lambda) x, x \rangle_{\mathcal{H}_0} \end{array} .$$

Then, for all x, $\lambda \mapsto f(\lambda, x)$ is measurable and, for all $\lambda \in \Lambda$, $x \mapsto f(\lambda, x)$ is continuous in x for the weak topology by Lemma 5.10. Moreover the correspondence

$$\varphi: \begin{array}{l} \Lambda \twoheadrightarrow \bar{B}_{0,1} \\ \lambda \mapsto \bar{B}_{0,1} \end{array}$$

is weakly measurable (in the sense of [1, Definition 18.1]) with nonempty compact values (for the weak topology). Therefore the Measurable Maximum Theorem [1, Theorem 18.19] gives that $m : \lambda \mapsto \max_{x \in \bar{B}_{0,1}} f(\lambda, x)$ is measurable and that there exists a function $g : \Lambda \to \bar{B}_{0,1}$ such that for all $\lambda \in \Lambda$, $g(\lambda) \in \operatorname{argmax}_{x \in \bar{B}_{0,1}} f(\lambda, x)$ and g is measurable from Λ to $\bar{B}_{0,1}$ endowed with the Borel σ-field generated by the weak topology. This implies the usual measurability by Lemma 5.9. We set $\sigma_0 = m$ and $\phi_0 = g$. Then, from the definitions of f, m and g, that $\sigma_0(\lambda)$ is the largest eigenvalue of $\Phi(\lambda)$ and that $\phi_0(\lambda)$ is an eigenvector with eigenvalue $\sigma_0(\lambda)$.

Construction of (σ_n, ϕ_n) **:** Assume we have constructed n measurable functions $\sigma_0, \cdots, \sigma_{n-1}$ and $\phi_0, \cdots, \phi_{n-1}$ satisfying for all $\lambda \in \Lambda$, $\sigma_0(\lambda) \geq \cdots \geq \sigma_{n-1}(\lambda)$, and $(\phi_0(\lambda), \cdots, \phi_{n-1}(\lambda))$ is an orthonormal family where for all $0 \leq i \leq n-1$, $\phi_i(\lambda) \in \ker(\Phi(\lambda) - \sigma_i(\lambda) \mathrm{Id}_{\mathcal{H}_0})$. Then, as in the initialization step, the function

$$f: \begin{array}{lcl} \Lambda \times \bar{B}_{0,1} & \to & \mathbb{R}_+ \\ (\lambda, x) & \mapsto & \langle \Phi(\lambda) x, x \rangle_{\mathcal{H}_0} - \sum_{i=1}^{n-1} \sigma_i(\lambda) \left| \langle x, \phi_i(\lambda) \rangle_{\mathcal{H}_0} \right|^2 \end{array} .$$

is measurable in λ and continuous in x (for the weak topology) by Lemma 5.10 and the correspondence

$$\varphi: \begin{array}{lcl} \Lambda & \twoheadrightarrow & \bar{B}_{0,1} \\ \lambda & \mapsto & \bar{B}_{0,1} \cap \operatorname{Span}(\phi_0(\lambda), \cdots, \phi_{n-1}(\lambda))^\perp \end{array}$$

is weakly measurable (in the sense of [1, Definition 18.1]) because of [1, Corollary 18.8 and Lemma 18.2]) and the fact that $\varphi(\lambda) = \left\{ x \in \bar{B}_{0,1} : \sum_{i=0}^{n-1} \left| \langle x, \phi_i(\lambda) \rangle_{\mathcal{H}_0} \right|^2 = 0 \right\}$ and has nonempty compact values (because $\varphi(\lambda)$ is a closed subset of $\bar{B}_{0,1}$ for the weak topology hence is compact for this topology). Hence, as previously, the Measurable Maximum Theorem and Lemma 5.9 give that $m : \lambda \mapsto \max_{x \in \varphi(\lambda)} f(\lambda, x)$ is measurable and that there exists a measurable function $g : \Lambda \to \mathcal{H}_0$ such that for all $\lambda \in \Lambda$, $g(\lambda) \in \operatorname{argmax}_{x \in \varphi(\lambda)} f(\lambda, x)$. We set $\sigma_n = m$ and $\phi_n = g$. Then, from the definitions of f, m and g, we get that $\sigma_n(\lambda) \leq \sigma_{n-1}(\lambda)$ is the $(n+1)$-th largest eigenvalue of $\Phi(\lambda)$ (because it is the largest eigenvalue of $\Phi(\lambda) - \sum_{i=0}^{n-1} \sigma_i(\lambda) \phi_i(\lambda) \otimes \phi_i(\lambda)$) and that $\phi_n(\lambda)$ is an eigenvector with eigenvalue $\sigma_n(\lambda)$ and is orthogonal to $\phi_0, \cdots, \phi_{n-1}$. □

We can now prove Lemma 4.9.

Proof of Lemma 4.9. We provide a proof in the case where $N = \infty$ as the finite dimensional case is easier. Let $f \in L^1(\Lambda, \mathcal{A}, \mathcal{S}_1^+(\mathcal{H}_0), \mu)$ be the density of ν with respect to μ. We assume without loss of generality that $f(\lambda) \in \mathcal{S}_1(\mathcal{H}_0)^+$ for all $\lambda \in \hat{\mathrm{G}}$ (rather than for μ-almost every λ). Using Theorem 5.11 we can write

$$f(\lambda) = \sum_{n=0}^{+\infty} \sigma_n(\lambda) \phi_n(\lambda) \otimes \phi_n(\lambda) , \tag{5.16}$$

where $(\sigma_n(\lambda))_{n \in \mathbb{N}}$ is non-decreasing and converges to zero and $(\phi_n(\lambda))_{n \in \mathbb{N}}$ satisfies (ii). Moreover, for all $\lambda \in \Lambda$, $\sum_n \sigma_n(\lambda) = \|f(\lambda)\|_1 < \infty$, and we get Assertions (i) and (iii).

It only remains to prove (iv)–(vi), which we now proceed to do. By (5.16) and the previously proved assertions, we get that for all $n \in \mathbb{N}$ and all $\lambda \in \Lambda$, $\left\|\phi_n^{\mathsf{H}} f^{1/2}(\lambda)\right\|_2^2 = \sigma_n(\lambda) \leq \|f(\lambda)\|_1$. Hence $\phi_n^{\mathsf{H}} f^{1/2} \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathbb{C}), \nu)$ and Proposition 3.4 gives that $\phi_n^{\mathsf{H}} \in \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathbb{C}), \nu)$ and for all $n, p \in \mathbb{N}$,

$$\left\langle \phi_n^{\mathsf{H}}, \phi_p^{\mathsf{H}} \right\rangle_\nu = \int \phi_n^{\mathsf{H}} f \phi_p \, \mathrm{d}\mu = \begin{cases} 0 & \text{if } n \neq p, \\ \int \sigma_n \, \mathrm{d}\mu & \text{otherwise. } , \end{cases}$$

where the last equality comes from (5.16) and the previously proved assertions.

Similarly, for all $n \in \mathbb{N}$, $\phi_n \otimes \phi_n f^{1/2} \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0), \nu)$, hence by Proposition 3.4, we have $\phi_n \otimes \phi_n \in \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0), \nu)$ and for all $n, p \in \mathbb{N}$,

$$\begin{aligned} [\phi_n \otimes \phi_n, \phi_p \otimes \phi_p]_\nu &= \int (\phi_n \otimes \phi_n) f (\phi_p \otimes \phi_p) \, \mathrm{d}\mu \\ &= \begin{cases} 0 & \text{if } n \neq p, \\ \int \sigma_n \, (\phi_n \otimes \phi_n) \, \mathrm{d}\mu & \text{otherwise,} \end{cases} \end{aligned}$$

which proves Assertion (v). Now observe that, for all $\lambda \in \Lambda$,

$$\left(\sum_{n=0}^{\infty} \phi_n(\lambda) \otimes \phi_n(\lambda) \right) f(\lambda) = f(\lambda) \left(\sum_{n=0}^{\infty} \phi_n(\lambda) \otimes \phi_n(\lambda) \right) = f(\lambda) \; .$$

This yields

$$\left\| \sum_{n=0}^{\infty} \phi_n \otimes \phi_n - \mathrm{Id}_{\mathcal{H}_0} \right\|_\nu = 0 \; ,$$

and thus Assertion (vi) holds, which concludes the proof. □

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