Weakly stationary stochastic processes valued in a separable Hilbert space: Gramian-Cramér representations and applications

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June 3, 2022

Abstract

The spectral theory for weakly stationary processes valued in a separable Hilbert space has known renewed interest in the past decade. Here we follow earlier approaches which fully exploit the *normal Hilbert module* property of the time domain. The key point is to build the Gramian-Cramér representation as an isomorphic mapping from the *modular spectral domain* to the *modular time domain*. We also discuss the general Bochner theorem and provide useful results on the composition and inversion of lag-invariant linear filters. Finally, we derive the Cramér-Karhunen-Loève decomposition and harmonic functional principal component analysis, which are established without relying on additional assumptions.

1 Introduction

Spectral theory for weakly stationary time series has been originally developed in a very general fashion, starting from the seminal works by Kolmogoroff, [17], and spanning over several decades, see [13] and the references therein. These foundations include time domain and frequency domain analyses, Cramér (or spectral) representations, the Herglotz theorem and linear filters. In [17, 13] the adopted framework is that of a bi-sequence $X = (X_t)_{t\in\mathbb{Z}} \in \mathcal{H}^{\mathbb{Z}}$ valued in a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ and weakly stationary in the sense that $\langle X_s, X_t\rangle_{\mathcal{H}}$ only depends on the lag $s-t$. In this framework, a linear filter is a linear operator on H_X onto H_X which commutes with the lag operator U^X, where H_X is the closure in $\mathcal{H}$ of the linear span of $(X_t)_{t\in\mathbb{Z}}$ and U^X is the operator defined on H_X by mapping X_t to X_{t+1} for all $t \in \mathbb{Z}$. As explained in [13, Section 3], a complete description of such a filter is given in the spectral domain by its transfer function. Let us recall the essential formulas which summarize what this means. In [13], the spectral theory follows from and start with the *canonical representation* of the lag operator U^X above, namely

$$U^X = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda}\,\xi(\mathrm{d}\lambda)\,, \tag{1.1}$$

where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and ξ is the spectral measure of U^X (which is a measure valued in the space of operators on H_X onto itself). This corresponds to [13, Eq. (8)] with a slightly different notation. Then defining $\hat{X}$ as $\xi(\cdot)X_0$ (thus a measure valued in H_X), one gets the celebrated Cramér representation (see [13, Eq. (13a)] again with a slightly different notation)

$$X_t = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda\,t}\,\hat{X}(\mathrm{d}\lambda)\,, \quad t \in \mathbb{Z}\,. \tag{1.2}$$

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Math Subject Classification. Primary: 60G12; Secondary: 47A56, 46G10
Keywords. Spectral representation of random processes. Isometries on Hilbert modules. Functional time series.

An other consequence of (1.1) is what is called the Herglotz theorem in [13, Eq. (9)], summarized by the formula

$$\langle X_s, X_t \rangle_{\mathcal{H}} = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda\,(s-t)}\ \mu(\mathrm{d}\lambda)\ , \quad s, t \in \mathbb{Z}\ , \tag{1.3}$$

where $\mu = \langle \xi(\cdot) X_0, X_0 \rangle_{\mathcal{H}}$ is a non-negative measure on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$. Interpreting the right-hand side of (1.3) as the scalar product of the two functions $\mathrm{e}_s : \lambda \mapsto \mathrm{e}^{\mathrm{i}\lambda s}$ and $\mathrm{e}_t : \lambda \mapsto \mathrm{e}^{\mathrm{i}\lambda t}$ in $L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu)$, Relation (1.3) is simply saying that the Cramér representation (1.2) mapping e_t to X_t is isometric. Following this interpretation, one can extend this isometric mapping to a unitary operator between the two isomorphic Hilbert spaces $L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu)$ and H_X, respectively refered to as the *spectral domain* and the *time domain*. In particular the output of a linear filter with transfer function $\Phi \in L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu)$ is given by

$$Y_t = \int \mathrm{e}^{\mathrm{i}\lambda t}\, \Phi(\lambda)\ \hat{X}(\mathrm{d}\lambda)\ , \quad t \in \mathbb{Z}\ , \tag{1.4}$$

or in other words, Y_t is the image of the function $\mathrm{e}_t \Phi$ by the extended unitary operator that maps the spectral domain to the time domain.

The spectral theory (1.1)–(1.4) applies to univariate times series by letting $\mathcal{H}$ be the space $L^2(\Omega, \mathcal{F}, \mathbb{P})$ of $\mathbb{C}$-valued random variables on $(\Omega, \mathcal{F})$ with finite second order moment. It also applies to multivariate time series by taking $\mathcal{H} = L^2(\Omega, \mathcal{F}, \mathbb{C}^q, \mathbb{P}) = (L^2(\Omega, \mathcal{F}, \mathbb{P}))^q$ and to functional time series by letting $\mathcal{H}$ be the Bochner space $L^2(\Omega, \mathcal{F}, L^2(0,1), \mathbb{P})$ of measurable mappings $V : \Omega \to L^2(0,1)$ such that $\mathbb{E}\left[\|V\|^2_{L^2(0,1)}\right] < \infty$, where $\|\cdot\|_{L^2(0,1)}$ here denotes the norm endowing the Hilbert space $L^2(0,1)$. However, in [13, Section 7], Holmes argues that important generalizations are needed for multivariate time series. This claim applies even more to functional time series. In this paper, we address such a generalization valid in the functional context. Related issued have been recently considered in [21, 22, 26] where, in particular, the authors derive a functional version of the Cramér representation which relies on a spectral density operator defined under strong assumptions on the covariance structure of the time series. Under the same assumption, [22] introduced filters whose transfer functions are valued in a restricted set of operators and this was latter generalized to bounded-operator-valued transfer functions in [26, Section 2.5] (see also [27, Appendix B.2.3]). An application of this spectral theory to dimension reduction is proposed by the means of a harmonic functional principal components analysis (see [22, 14]). A more general approach is adopted in [28] where the authors provide a definition of operator-valued measures from which they derive a functional version of the Herglotz theorem, the functional Cramér representation, the definition of linear filters with bounded-operator-valued transfer functions and a harmonic functional principal component analysis in the case where the spectral measure has finitely many discontinuities.

To complement these recent works, we here focus on the Gramian structure of the space $\mathcal{H} = L^2(\Omega, \mathcal{F}, L^2(0,1), \mathbb{P})$. This approach extends naturally the results gathered in [20] for the multivariate case and where the Gramian nature of the covariance matrix plays a key role. In particular, in the multivariate case, the lag operator U^X is not only (scalar product) isometric on H_X but also Gramian-isometric on the larger space $\overline{\mathrm{Span}}\left(\mathrm{Q} X_t\ ,\ t \in \mathbb{Z}, \mathrm{Q} \in \mathbb{C}^{q \times q}\right)$. In the functional case, we exhibit the Gramian structure of $\mathcal{H} = L^2(\Omega, \mathcal{F}, L^2(0,1), \mathbb{P})$ by making it a *normal Hilbert module*. As a result, the time domain space H_X of [13] is replaced by the *modular time domain*

$$\mathcal{H}^X = \overline{\mathrm{Span}}\left(\mathrm{Q} X_t\ ,\ t \in \mathbb{Z},\ \mathrm{Q} \in \mathcal{L}_b(L^2(0,1))\right)\ , \tag{1.5}$$

where $\mathcal{L}_b(L^2(0,1))$ denotes the space of bounded operators on $L^2(0,1)$ onto itself. In comparison, in the definition of H_X used in [13], Q is restricted to be a scalar operator. Thus, while H_X is a subspace of $\mathcal{H}$ seen as a Hilbert space, $\mathcal{H}^X$ is a submodule of $\mathcal{H}$ seen as a normal Hilbert module. Based on this simple fact, a natural path for achieving and fully exploiting a Cramér representation on $\mathcal{H}^X$ is:

Step 1) Interpret the representation (1.1) as the one of a Gramian-isometric operator on $\mathcal{H}^X$ (and not only an scalar product isometric operator on H_X).

Step 2) Deduce that the Cramér representation (1.2) can effectively be extended as a Gramian-isometric operator mapping $L^2(0,1) \to L^2(0,1)$-operator-valued functions on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ to an element of $\mathcal{H}^X$.

Step 3) As a first consequence, the scalar product isometric relation (1.3) is extended to

$$[X_s, X_t]_{\mathcal{H}} = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda\,(s-t)}\ \nu(\mathrm{d}\lambda)\ , \quad s, t \in \mathbb{Z}\ , \tag{1.6}$$

where, here, ν is an operator-valued measure on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ and $[X_s, X_t]_{\mathcal{H}} = \mathrm{Cov}\,(X_s, X_t)$. This Gramian-isometric relationship corresponds to what is called the Herglotz theorem in the functional time series case.

Step 4) As a second consequence, the Cramér representation (1.4) of a linear filter is extended to the case where the transfer function Φ is now an $L^2(0,1) \to L^2(0,1)$-operator-valued functions on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ (and not only a scalar valued functions on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$). This raises the question, in particular, of the precise condition required on the transfer function to replace the condition $\Phi \in L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu)$ of the scalar case.

Step 5) An interesting consequence of Step 4) is to study the composition of linear filters and deduce when and how it is possible to invert them.

Step 6) An other interesting consequence of Step 2) is to derive the Cramér-Karhunen-Loève decomposition and the harmonic principal component analysis for any weakly stationary functional time series valued in a separable Hilbert space.

In this contribution, we basically follow this path, up to the following slight modifications.

1. We treat the more general case of a stochastic process $(X_t)_{t\in\mathrm{G}}$, where $(\mathrm{G}, +)$ is a locally compact Abelian (l.c.a.) group set of indices and for each $t \in \mathrm{G}$, X_t is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and valued in a separable Hilbert space $\mathcal{H}_0$ (endowed with its Borel σ-field). Typical examples for G and $\mathcal{H}_0$ are the ones of functional time series, namely $\mathrm{G} = \mathbb{Z}$ and $\mathcal{H}_0 = L^2(0,1)$ but, as far as spectral theory is concerned, the presentation of the results is not only more general (one can *e.g.* take $\mathrm{G} = \mathbb{R}$) but also more elegant in this general setting. Of course, in the discrete time case $\mathrm{G} = \mathbb{Z}$, any continuity condition imposed on a function defined on G is trivially satisfied. Such continuity conditions constitute a small price to pay (and the only one) in order to be able to treat the case of a general l.c.a. group G rather than focusing on the discrete time case alone.
2. For obvious practical reasons, it is usual to treat the mean of a stochastic process separately. Therefore we will assume that the process $(X_t)_{t\in\mathrm{G}}$ is centered.
3. We will consider the case where the separable Hilbert space $\mathcal{G}_0$ in which the output of the filter is valued is different from $\mathcal{H}_0$, the one of the input, that is, we replace $\mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0)$ in (1.5) by $\mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, the space of bounded operators from $\mathcal{H}_0$ to $\mathcal{G}_0$. This makes the results directly applicable in the case of different input and output spaces, especially in the case where they have different dimensions (so that they are not isomorphic).

The approach to derive a spectral theory following Step 1)– Step 4) is essentially contained in [16, 19, 15]. Our main contribution concerning these steps is to introduce all the preliminary definitions required to understand them, to select the most important results, to provide detailed proofs of the key points and to bring forward this approach which offers an interesting alternative to the ones recently proposed in [21, 22, 26, 27, 28]. A first benefit of the Gramian-isometric approach is that it allows a concrete description of the spectral domain rather than relying on the completion of a pre-Hilbert space or on the compactification of a pointed convex cone as used in [26, Section 2.5] and [28], respectively. A second benefit is to make the Cramér representation much easier to exploit for deriving useful general results. This will be made apparent when establishing the composition and inversion of filters of Step 5), which to our best knowledge, appear to be novel in this degree of generality. Similarly, our versions of the Cramér-Karhunen-Loève decomposition and harmonic functional principal component analysis are not restricted to the case where the spectral density operator is continuous or the spectral measure has finitely many discontinuities as in [26, 28]. However, it is important to note that, contrary to [21, 22, 26, 14], we do not address the question of statistical estimation in the spectral domain. The spectral theory we present applies to all weakly stationary processes whereas statistical inference results require specific assumptions. As in the univariate setting, the spectral analysis of long memory processes necessitates assumptions and technical developments beyond the ones used for the spectral analysis of short memory processes. In the functional setting, up to our knowledge, long memory processes have been mostly studied in the time domain (see e.g. [24, 5, 6, 12, 18]). Clarifying the general spectral theory that applies to all functional weakly stationary processes is a first step towards studying functional long memory processes in the spectral domain as classically done for univariate long memory processes (see e.g. [23, Section 2.4] about the celebrated FARIMA processes). Such a study, however, is out of the scope of the present paper.

The paper is organized as follows. Basic definitions of operator-valued measures, operator-valued functions (and the various notions of measurability related to them) and Gramian-isometric operators on normal Hilbert modules are assembled in Section 2. Section 3 contains some preliminaries paving the way for describing the *modular spectral domain*. In particular, we explain how to use normal Hilbert modules for defining Gramian-orthogonally scattered measures. Section 4 contains the main results: 1) we offer a synthesis of the results of [16, 19, 15] providing a natural and complete spectral theory for weakly stationary processes valued in a separable Hilbert space; 2) then, this approach is exploited to address Step 5) and Step 6) above, successively. All the proofs are postponed in Section 5 along with additional useful results.

Contents

2 Basic definitions and notation

2.1 Operators, measurability and integrals

Basic definitions on linear operators can be found, for example, in [29] and we refer the reader to [11, Chapter 1] for a nice overview of measurability and integration on Banach spaces. Throughout this paper, we will denote by $\mathcal{O}(\mathcal{H}, \mathcal{G})$ the set of linear operators Q from the (complex) Hilbert space $\mathcal{H}$ to the (complex) Hilbert space $\mathcal{G}$ whose domains, denoted by $\mathcal{D}(\mathrm{Q})$, are linear subspaces of $\mathcal{H}$. We then denote by $\mathcal{L}_b(\mathcal{H}, \mathcal{G})$ its subset of continuous operators, by $\mathcal{K}(\mathcal{H}, \mathcal{G})$ its subset of compact continuous operators and, for all $p \in [1, \infty)$, by $\mathcal{S}_p(\mathcal{H}, \mathcal{G})$ the Schatten-p class of compact operators with ℓ^p singular values. Schatten-1 and Schatten-2 operators are usually referred to as *trace-class* and *Hilbert-Schmidt* operators respectively. If $\mathcal{G} = \mathcal{H}$, we omit $\mathcal{G}$ in the notation of these operator sets. For $\mathrm{Q} \in \mathcal{L}_b(\mathcal{H}, \mathcal{G})$ we denote its adjoint by Q^{H}. We denote by $\mathcal{L}_b^+(\mathcal{H})$ the set of *positive* operators i.e. the set of $\mathrm{Q} \in \mathcal{L}_b(\mathcal{H})$ such that $\langle \mathrm{Q}x, x \rangle_{\mathcal{H}} \geq 0$ for all $x \in \mathcal{H}$. Similarly, $\mathcal{K}^+(\mathcal{H})$ and $\mathcal{S}_p^+(\mathcal{H})$ denote respectively the sets of positive compact and positive Schatten-p operators. If $\mathrm{Q} \in \mathcal{K}^+(\mathcal{H})$, $\mathrm{Q}^{1/2}$ denote the unique operator of $\mathcal{K}^+(\mathcal{H})$ which satisfies $\mathrm{Q} = \left(\mathrm{Q}^{1/2}\right)^2$. The notation $\|\cdot\|$ is used for the operator norm on $\mathcal{L}_b(\mathcal{H}, \mathcal{G})$ and $\|\cdot\|_p$ is used for the Schatten-p norm on $\mathcal{S}_p(\mathcal{H}, \mathcal{G})$. For $E \subset \mathcal{H}$, we will used the notation $\overline{\mathrm{Span}}^{\mathcal{H}}(E)$ for the smallest linear subspace of $\mathcal{H}$ which contains E and is closed for the norm topology in $\mathcal{H}$.

For a measurable space $(\Lambda, \mathcal{A})$ and a Banach space $(E, \|\cdot\|_E)$, we denote by $\mathbb{F}(\Lambda, \mathcal{A}, E)$ the space of measurable functions from $(\Lambda, \mathcal{A})$ to $(E, \mathcal{B}(E))$, where $\mathcal{B}(E)$ is the Borel σ-field on E. For a non-negative measure μ on $(\Lambda, \mathcal{A})$ and $p \in [1, \infty]$, we denote by $\mathcal{L}^p(\Lambda, \mathcal{A}, E, \mu)$ the space of functions $f \in \mathbb{F}(\Lambda, \mathcal{A}, E)$ such that $\int \|f\|_E^p \, \mathrm{d}\mu$ (or μ-essup $\|f\|_E$ for $p = \infty$) is finite and by $L^p(\Lambda, \mathcal{A}, E, \mu)$ its quotient space with respect to μ-a.e. equality. The corresponding norm is denoted by $\|\cdot\|_{L^p(\Lambda, \mathcal{A}, E, \mu)}$. The Bochner integral is defined on $L^1(\Lambda, \mathcal{A}, E, \mu)$ by linear and continuous extension of the mapping $\mathbb{1}_A x \to \mu(A)\, x$ defined for $x \in E$ and $A \in \mathcal{A}$ such that $\mu(A) < \infty$. In the particular case where E is a space of linear operators between two Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$, we use the following weaker notion of measurability.

Definition 2.1 (Simple measurability)**.** *A function $\Phi : \Lambda \to \mathcal{L}_b(\mathcal{H}, \mathcal{G})$ is said to be* simply measurable *if for all $x \in \mathcal{H}$, $\lambda \mapsto \Phi(\lambda)x$ is measurable as a $\mathcal{G}$-valued function. The set of such functions is denoted by $\mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}, \mathcal{G})$ or simply $\mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H})$ if $\mathcal{G} = \mathcal{H}$.*

Note that for all Banach spaces $\mathcal{E}$ which are continuously embedded in $\mathcal{L}_b(\mathcal{H}, \mathcal{G})$ (*e.g.* $\mathcal{S}_p(\mathcal{H}, \mathcal{G})$ for $p \geq 1$ or $\mathcal{K}(\mathcal{H}, \mathcal{G})$), the following inclusions hold

$$\mathbb{F}(\Lambda, \mathcal{A}, \mathcal{E}) \subseteq \mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}, \mathcal{G}) \ . \tag{2.1}$$

When $\mathcal{H}$ and $\mathcal{G}$ are separable, the equality holds for $\mathcal{E} = \mathcal{K}(\mathcal{H}, \mathcal{G})$ and for $\mathcal{E} = \mathcal{S}_p(\mathcal{H}, \mathcal{G})$ with $p \in \{1, 2\}$, see Lemma 5.1.

2.2 Vector-valued and Positive Operator-Valued Measures

Measures valued in a Banach space, and in particular Positive Operator-Valued Measures, are key in the spectral theory of functional times series. This section gathers results on such measures. Details can be found in [10, 2]. First, we recall that a measure μ defined on the measurable space $(\Lambda, \mathcal{A})$ and valued in the Banach space $(E, \|\cdot\|_E)$ is an $\mathcal{A} \to E$ mapping such that, for any sequence $(A_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ of pairwise disjoint sets, $\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$, where the series converges in E, that is,

$$\lim_{N \to +\infty} \left\| \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) - \sum_{n=0}^{N} \mu(A_n) \right\|_E = 0 \ . \tag{2.2}$$

For such a measure μ, the mapping

$$\|\mu\|_E : A \mapsto \sup \left\{ \sum_{i \in \mathbb{N}} \|\mu(A_i)\|_E \ : \ (A_i)_{i \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}} \text{ is a countable partition of } A \right\}$$

defines a non-negative measure on $(\Lambda, \mathcal{A})$ called the *variation measure* of μ. For instance, if $E = \mathcal{S}_1(\mathcal{H})$, we write $\|\mu\|_1$ since we use $\|\cdot\|_1$ to denote the Schatten-1 norm. Integrals of functions in $L^1(\Lambda, \mathcal{A}, \|\mu\|_E)$ with respect to μ are discussed in [10, P. 120]. When Λ is a locally compact topological space and $\mathcal{A}$ is the Borel σ-field, an E-valued measure μ is said to be *regular* if for all $A \in \mathcal{A}$ and $\epsilon > 0$, there exist a compact set $K \in \mathcal{A}$ and an open set $U \in \mathcal{A}$ with $K \subset A \subset U$ such that $\|\mu(U \setminus K)\|_E \leq \epsilon$. The special case of operator-valued measures is of particular interest to us and, specifically, Positive Operator-Valued Measures (p.o.v.m.'s). We recall that a sequence $(\mathrm{Q}_n)_{n \in \mathbb{N}} \in \mathcal{L}_b(\mathcal{H})^{\mathbb{N}}$ converges to an operator $\mathrm{Q} \in \mathcal{L}_b(\mathcal{H})$ in *weak operator topology* (w.o.t.) if for all $x \in \mathcal{H}$, $\lim_{n \to \infty} \langle \mathrm{Q}_n x, x \rangle_{\mathcal{H}} = \langle \mathrm{Q} x, x \rangle_{\mathcal{H}}$.

Definition 2.2 (Positive Operator-Valued Measures (p.o.v.m.))**.** *Let $(\Lambda, \mathcal{A})$ be a measurable space and $\mathcal{H}$ be a Hilbert space. A Positive Operator-Valued Measure (p.o.v.m.) on $(\Lambda, \mathcal{A}, \mathcal{H})$ is a mapping $\nu : \mathcal{A} \to \mathcal{L}_b^+(\mathcal{H})$ such that for all sequences of disjoint sets $(A_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$,*

$$\nu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \nu(A_n) \tag{2.3}$$

where the series converges in $\mathcal{L}_b^+(\mathcal{H})$ in w.o.t.

Note that the series in (2.3) does not necessarily converge in operator norm which implies that, in this definition, a p.o.v.m. does not need to be an $\mathcal{L}_b(\mathcal{H})$-valued measure in the sense of (2.2). Therefore the above definitions of integrals and regularity cannot be applied. This is circumvented by noting that a p.o.v.m. is entirely characterized by the family of non-negative measures $\left\{\nu_x : A \mapsto x^{\mathsf{H}}\nu(A)x \,:\, x \in \mathcal{H}\right\}$. We refer to Definition 14 and Theorem 20 in [2] for details about regular p.o.v.m.'s and to Theorem 9 in [2] for details about integration of bounded scalar functions with respect to a p.o.v.m.

When dealing with spectral operator measures of weakly stationary processes valued in a separable Hilbert space, we can rely on the additional trace-class property, which makes all the previous definitions easier to handle and extend.

Definition 2.3 (Trace-class p.o.v.m.). *Let $(\Lambda, \mathcal{A})$ be a measurable space, $\mathcal{H}_0$ be a* separable *Hilbert space and ν be a p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. We say that ν is a* trace-class-*p.o.v.m. if it is $\mathcal{S}_1^+(\mathcal{H}_0)$-valued.*

The first advantage of a trace-class p.o.v.m. is that it fits the framework of vector-valued measures, namely, we have the following result, whose proof can be found in Section 5.1.

Lemma 2.1. *Let $(\Lambda, \mathcal{A})$ be a measurable space and $\mathcal{H}_0$ be a* separable *Hilbert space. Then a p.o.v.m. ν on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ is trace-class if and only if $\nu(\Lambda) \in \mathcal{S}_1(\mathcal{H}_0)$. In this case, ν is an $\mathcal{S}_1(\mathcal{H}_0)$-valued measure (in the sense that* (2.3) *holds in $\|\cdot\|_1$-norm) with finite variation measure $\|\nu\|_1 : A \mapsto \|\nu(A)\|_1$. Moreover, ν is regular if and only if $\|\nu\|_1$ is regular.*

Another advantage of trace-class p.o.v.m.'s is that they satisfy the Radon-Nikodym property. Namely, if ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ and μ is a σ-finite non-negative measure on $(\Lambda, \mathcal{A})$, then $\|\nu\|_1 \ll \mu$ (*i.e.* for all $A \in \mathcal{A}$, $\mu(A) = 0 \Rightarrow \|\nu\|_1(A) = 0$), if and only if there exists $g \in L^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{H}_0), \mu)$ such that $\mathrm{d}\nu = g\,\mathrm{d}\mu$, *i.e.* for all $A \in \mathcal{A}$,

$$\nu(A) = \int_A g\,\mathrm{d}\mu\,. \tag{2.4}$$

In this case, g is unique and is called the density of ν with respect to μ and we write $g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. This result is a consequence of Theorem 1 in [9, Chapter III, Section 3] because $\mathcal{S}_1(\mathcal{H}_0)$ is the dual of the separable space $\mathcal{K}(\mathcal{H}_0)$.

2.3 Normal Hilbert modules

Modules extend the notion of vector spaces to the case where scalar multiplication is replaced by a multiplicative operation with elements of a ring. The case where the ring is $\mathcal{L}_b(\mathcal{H}_0)$ for a separable Hilbert space $\mathcal{H}_0$ is of particular interest for $\mathcal{H}_0$-valued random variables. In short, a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module is a Hilbert space endowed with a *module action* and a *Gramian*. A Gramian $[\cdot,\cdot]$ is similar to a scalar product but is valued in the space $\mathcal{S}_1(\mathcal{H}_0)$ and is related to scalar product by the relation $\langle\cdot,\cdot\rangle = \mathrm{Tr}([\cdot,\cdot])$. Notions such as sub-modules, Gramian-orthogonality, Gramian-isometric operators are natural extensions of their counterparts in the Hilbert framework. We give such useful definitions hereafter and refer to [15, Chapter 2] for details.

Definition 2.4 ($\mathcal{L}_b(\mathcal{H}_0)$-module). *Let $\mathcal{H}_0$ be a separable Hilbert space. An $\mathcal{L}_b(\mathcal{H}_0)$-module is a commutative group $(\mathcal{H}, +)$ such that there exists a multiplicative operation (called the* module action*)*

$$\begin{array}{ccc} \mathcal{L}_b(\mathcal{H}_0) \times \mathcal{H} & \to & \mathcal{H} \\ (\mathrm{Q}, x) & \mapsto & \mathrm{Q} \bullet x \end{array}$$

which satisfies the usual distributive properties : for all $\mathrm{Q}, \mathrm{T} \in \mathcal{L}_b(\mathcal{H}_0)$, and $x, y \in \mathcal{H}$,

$$\begin{aligned} \mathrm{Q} \bullet (x + y) &= \mathrm{Q} \bullet x + \mathrm{Q} \bullet y, \\ (\mathrm{Q} + \mathrm{T}) \bullet x &= \mathrm{Q} \bullet x + \mathrm{T} \bullet x, \\ (\mathrm{QT}) \bullet x &= \mathrm{Q} \bullet (\mathrm{T} \bullet x), \\ \mathrm{Id}_{\mathcal{H}_0} \bullet x &= x. \end{aligned}$$

Next, we endow an $\mathcal{L}_b(\mathcal{H}_0)$-module with an $\mathcal{L}_b(\mathcal{H}_0)$-valued product.

Definition 2.5 ((Normal) pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space. We say that $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{H}})$ is a pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module if $\mathcal{H}$ is an $\mathcal{L}_b(\mathcal{H}_0)$-module and $[\cdot,\cdot]_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathcal{L}_b(\mathcal{H}_0)$ satisfies, for all $x, y, z \in \mathcal{H}$, and $\mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0)$,*

(i) $[x, x]_{\mathcal{H}} \in \mathcal{L}_b^+(\mathcal{H}_0)$,

(ii) $[x, x]_{\mathcal{H}} = 0$ *if and only if* $x = 0$,

(iii) $[x + \mathrm{Q} \bullet y, z]_{\mathcal{H}} = [x, z]_{\mathcal{H}} + \mathrm{Q}[y, z]_{\mathcal{H}}$,

(iv) $[y, x]_{\mathcal{H}} = [x, y]_{\mathcal{H}}^{\mathsf{H}}$.

If moreover, for all $x, y \in \mathcal{H}$, $[x, y]_{\mathcal{H}} \in \mathcal{S}_1(\mathcal{H}_0)$, we say that $[\cdot,\cdot]_{\mathcal{H}}$ is a Gramian *and that $\mathcal{H}$ is a* normal *pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module.*

Note that an $\mathcal{L}_b(\mathcal{H}_0)$-module is a vector space if we define the scalar-vector multiplication by $\alpha x = (\alpha \mathrm{Id}_{\mathcal{H}_0}) \bullet x$ for all $\alpha \in \mathbb{C}$, $x \in \mathcal{H}$ and that, in the particular case where $[\cdot,\cdot]_{\mathcal{H}}$ is a Gramian, then $\langle \cdot,\cdot \rangle_{\mathcal{H}} := \mathrm{Tr}[\cdot,\cdot]_{\mathcal{H}}$ is a scalar product. Hence a normal pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module is also a pre-Hilbert space. A normal pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module is said to be a *normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module* if it is complete (for the norm defined by $\|x\|_{\mathcal{H}}^2 = \langle x, x \rangle_{\mathcal{H}} = \left\| [x, x]_{\mathcal{H}} \right\|_1$). A subset of $\mathcal{H}$ is called a *submodule* if it is an $\mathcal{L}_b(\mathcal{H}_0)$-module. An operator $\mathrm{F} \in \mathcal{L}_b(\mathcal{H}, \mathcal{G})$, where $\mathcal{H}$ and $\mathcal{G}$ are two $\mathcal{L}_b(\mathcal{H}_0)$-module, is said to be $\mathcal{L}_b(\mathcal{H}_0)$*-linear* if for all $\mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0)$ and $x \in \mathcal{H}$, $\mathrm{F}(\mathrm{Q} \bullet x) = \mathrm{Q} \bullet (\mathrm{F}x)$. An $\mathcal{L}_b(\mathcal{H}_0)$-linear operator U between two pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-modules $\mathcal{H}$ and $\mathcal{G}$ is said to be *Gramian-isometric* if for all $x, y \in \mathcal{H}$, $[Ux, Uy]_{\mathcal{G}} = [x, y]_{\mathcal{H}}$ and *Gramian-unitary* if it is bijective Gramian-isometric. The space $\mathcal{H}$ is said to be *Gramian-isometrically embedded* in $\mathcal{G}$ (denoted by $\mathcal{H} \underset{\sim}{\subseteq} \mathcal{G}$) if there exists a Gramian-isometric operator from $\mathcal{H}$ to $\mathcal{G}$. The spaces $\mathcal{H}$ and $\mathcal{G}$ are said to be *Gramian-isometrically isomorphic* (denoted by $\mathcal{H} \cong \mathcal{G}$) if there exists a Gramian-unitary operator from $\mathcal{H}$ to $\mathcal{G}$. The well known isometric extension theorem can be straightforwardly generalized to the case of Gramian-isometric operators and is stated in the following proposition for latter reference.

Proposition 2.2 (Gramian-isometric extension)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space, $\mathcal{H}$ be a normal pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module, and $\mathcal{G}$ be a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module. Let $(v_j)_{j \in J}$ and $(w_j)_{j \in J}$ be two collections of vectors in $\mathcal{H}$ and $\mathcal{G}$ respectively with J an arbitrary index set. If for all $i, j \in J$, $[v_i, v_j]_{\mathcal{H}} = [w_i, w_j]_{\mathcal{G}}$ then there exists a unique Gramian-isometric operator*

$$S : \overline{\mathrm{Span}}^{\mathcal{H}} \left(\mathrm{Q} \bullet v_j, \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0), j \in J\right) \to \mathcal{G}$$

such that for all $j \in J$, $S v_j = w_j$. If moreover $\mathcal{H}$ is complete then

$$S\left(\overline{\mathrm{Span}}^{\mathcal{H}} \left(\mathrm{Q} \bullet v_j, \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0), j \in J\right)\right) = \overline{\mathrm{Span}}^{\mathcal{G}} \left(\mathrm{Q} \bullet w_j, \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0), j \in J\right)$$

As stated in the introduction, the spectral theory for functional time series relies on the Gramian structure of the space of functional random variables with finite second order moment. The following example exhibits this structure.

Example 2.1 (Normal Hilbert module $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{H}_0$ be a separable Hilbert space. The Bochner space $L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ is the space of $\mathcal{H}_0$-valued random variables Y such that $\mathbb{E}\left[\|Y\|_{\mathcal{H}_0}^2\right] < +\infty$. Then the expectation of Y is the unique vector $\mathbb{E}[Y] \in \mathcal{H}_0$ satisfying*

$$\langle \mathbb{E}[Y], x \rangle_{\mathcal{H}_0} = \mathbb{E}\left[\langle Y, x \rangle_{\mathcal{H}_0}\right], \quad \text{for all } x \in \mathcal{H}_0 ,$$

and the covariance operator between $Y, Z \in L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ is the unique linear operator $\mathrm{Cov}(Y, Z) \in \mathcal{L}_b(\mathcal{H}_0)$, satisfying

$$\langle \mathrm{Cov}(Y, Z)\, y, x \rangle_{\mathcal{H}_0} = \mathrm{Cov}\left(\langle Y, x \rangle_{\mathcal{H}_0}, \langle Z, y \rangle_{\mathcal{H}_0}\right), \quad \text{for all } x, y \in \mathcal{H}_0 .$$

The space $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ of all centered random variables in $L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ is a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module for the module action defined for all $\mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0)$ and $X \in \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ by $\mathrm{Q} \bullet X = \mathrm{Q}X$, and the Gramian

$$[X, Y]_{\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})} = \mathrm{Cov}(X, Y) .$$

3 Towards the stochastic integral

3.1 Gramian-orthogonally scattered (g.o.s.) measures

In this section, we introduce the notion of random g.o.s. measures which will have an important role in the construction provided by [16, 19, 15]. The terminologies o.s. and g.o.s. are borrowed from Definition 3 in [15, Section 3.1]

Definition 3.1 ((Random) o.s. measures)**.** *Let $\mathcal{H}$ be a Hilbert space and $(\Lambda, \mathcal{A})$ be a measurable space. We say that $W : \mathcal{A} \to \mathcal{H}$ is a* countably additive orthogonally scattered *(o.s.) measure on $(\Lambda, \mathcal{A}, \mathcal{H})$ if it is an $\mathcal{H}$-valued measure on $(\Lambda, \mathcal{A})$ such that for all $A, B \in \mathcal{A}$,*

$$A \cap B = \emptyset \Rightarrow \langle W(A), W(B) \rangle_{\mathcal{H}} = 0 \, .$$

In this case, the mapping

$$\nu_W : A \mapsto \langle W(A), W(A) \rangle_{\mathcal{H}}$$

is a finite non-negative measure on $(\Lambda, \mathcal{A})$ called the intensity measure *of W and we have that, for all $A, B \in \mathcal{A}$,*

$$\nu_W(A \cap B) = \langle W(A), W(B) \rangle_{\mathcal{H}} \, . \tag{3.1}$$

We say that W is regular if ν_W is regular. When $\mathcal{H}$ is the space $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ of Example 2.1, we say that W is an $\mathcal{H}_0$-valued random o.s. measure *on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$.*

The generalization to a normal Hilbert module is straightforward.

Definition 3.2 ((Random) g.o.s. measures)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space, $\mathcal{H}$ be a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module and $(\Lambda, \mathcal{A})$ be a measurable space. We say that $W : \mathcal{A} \to \mathcal{H}$ is a* countably additive Gramian-orthogonally scattered *(g.o.s.) measure on $(\Lambda, \mathcal{A}, \mathcal{H})$ if it is an $\mathcal{H}$-valued measure on $(\Lambda, \mathcal{A})$ such that for all $A, B \in \mathcal{A}$,*

$$A \cap B = \emptyset \Rightarrow [W(A), W(B)]_{\mathcal{H}} = 0 \, .$$

In this case, the mapping

$$\nu_W : A \mapsto [W(A), W(A)]_{\mathcal{H}}$$

is a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ called the intensity operator measure *of W and we have that, for all $A, B \in \mathcal{A}$,*

$$\nu_W(A \cap B) = [W(A), W(B)]_{\mathcal{H}} \, . \tag{3.2}$$

We say that W is regular if $\|\nu_W\|_1$ is regular. When $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ of Example 2.1, we say that W is an $\mathcal{H}_0$-valued random g.o.s. measure *on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$.*

It is easy to show that a o.s. measure W as in Definition 3.1 can be equivalently seen as the restriction of an isometric operator I from $L^2(\Lambda, \mathcal{A}, \nu_W)$ onto $\mathcal{H}$ by setting

$$W(A) = I(\mathbb{1}_A) \, , \qquad A \in \Lambda \, .$$

This simply follows by interpreting the left-hand side of (3.1) as the scalar product between $\mathbb{1}_A$ and $\mathbb{1}_B$ in $L^2(\Lambda, \mathcal{A}, \nu_W)$ so that I above can be defined as the unique isometric extension from $L^2(\Lambda, \mathcal{A}, \nu_W)$ to $\mathcal{H}$ of the isometric mapping defined by $\mathbb{1}_A \mapsto W(A)$ for $A \in \Lambda$. This observation gives a rigorous meaning to the integral in the Cramér representation (1.2) where $\hat{X}$ is o.s. (see [13, Section 2]). Similarly, if W is a g.o.s. measure as in Definition 3.2 and $\mathcal{H}_0 = \mathbb{C}^q$, the mapping defined by $\mathbb{1}_A \mathrm{Q} \mapsto \mathrm{Q} W(A)$ for $A \in \Lambda$ and $\mathrm{Q} \in \mathbb{C}^{q \times q}$ is Gramian-isometric from a normal pre-Hilbert module of matrix-valued functions onto $\mathcal{H}$ (see [20]). This observation is a key step to derive a Cramér representation of the type (1.2) where $(X_t)_{t \in \mathbb{Z}}$ is a multivariate time series and $\hat{X}$ is g.o.s. In the infinite dimensional case, the Gramian-isometric property of the mapping defined by $\mathbb{1}_A \mathrm{Q} \mapsto \mathrm{Q} W(A)$ for $A \in \Lambda$ and $\mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0)$ can also be established. This is done in [26, Section 2.5] where the author uses the completion of $L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0), \|\nu\|_1)$ under an appropriate norm. These ideas are in fact very similar to the ones of [16, 19, 15] with the exception that the latter references provide a more general framework and lead to a modular spectral domain which is an explicit set of operator-valued functions defined on Λ. We follow this approach in the next section.

3.2 The space $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$

As discussed in the previous sections, the role of o.s. and g.o.s. measures in the spectral theory of weakly stationary processes relies on their characterization by unitary or Gramian-unitary operators between the (modular) time domain and the (modular) spectral domain. This has been entirely studied in the case of univariate and multivariate time series, see [13] and [20], respectively, and the references therein. For time series valued in a general separable Hilbert space, defining the modular spectral domain requires to exhibit a suitable space of operator-valued functions which are *square-integrable* with respect to the trace-class p.o.v.m. ν. It was introduced in [19] and includes the space $L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ but is in general larger in the case where $\mathcal{H}_0$ has infinite dimension. The definition relies on the following notion of measurability which we slightly adapted from [19], [15, Section 3.4].

Definition 3.3 ($\mathcal{O}$-measurability)**.** *Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$, a function $\Phi : \Lambda \to \mathcal{O}(\mathcal{H}, \mathcal{G})$ is said to be $\mathcal{O}$-measurable if it satisfies the two following conditions.*

(i) For all $x \in \mathcal{H}$, $\{\lambda \in \Lambda : x \in \mathcal{D}(\Phi(\lambda))\} \in \mathcal{A}$.

(ii) There exists a sequence $(\Phi_n)_{n\in\mathbb{N}}$ valued in $\mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}, \mathcal{G})$ such that for all $\lambda \in \Lambda$ and $x \in \mathcal{D}(\Phi(\lambda))$, $\Phi_n(\lambda)x$ converges to $\Phi(\lambda)x$ in $\mathcal{G}$ as $n \to \infty$.

We denote by $\mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}, \mathcal{G})$ the space of such functions Φ.

Square-integrability with respect to a trace-class p.o.v.m. ν is then defined as follows.

Definition 3.4. *Let $(\Lambda, \mathcal{A})$ be a measurable space, $\mathcal{H}_0, \mathcal{G}_0, \mathcal{I}_0$ be three separable Hilbert spaces and ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ with density f with respect to its finite variation $\|\nu\|_1$. Then, we say that $(\Phi, \Psi) \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0) \times \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{I}_0)$ is* ν-integrable *if the three following assertions hold.*

(i) We have $\mathrm{Im}(f^{1/2}) \subset \mathcal{D}(\Phi)$ and $\mathrm{Im}(f^{1/2}) \subset \mathcal{D}(\Psi)$, $\|\nu\|_1$-a.e.

(ii) We have $\Phi f^{1/2} \in \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$ and $\Psi f^{1/2} \in \mathcal{S}_2(\mathcal{H}_0, \mathcal{I}_0)$, $\|\nu\|_1$-a.e.

(iii) We have $(\Phi f^{1/2})(\Psi f^{1/2})^{\mathsf{H}} \in \mathcal{L}^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{I}_0, \mathcal{G}_0), \|\nu\|_1)$.

In this case, we define

$$\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} := \int (\Phi f^{1/2})(\Psi f^{1/2})^{\mathsf{H}} \, \mathrm{d}\|\nu\|_1 \in \mathcal{S}_1(\mathcal{I}_0, \mathcal{G}_0) \,. \tag{3.3}$$

Moreover, we say that $\Phi \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$ is square ν-integrable *if (Φ, Φ) is ν-integrable and we denote by $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ the space of square ν-integrable functions in $\mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$.*

Remark 3.1. *Let us briefly comment this definition.*

1) The integral (3.3) of Definition 3.4 can be seen as an extension of the integral of scalar-valued functions with respect to a trace-class p.o.v.m. since, for a measurable scalar function $\phi : \Lambda \to \mathbb{C}$ we can interpret the integral $\int \phi \, \mathrm{d}\nu$ as the one in (3.3) with $\Phi : \lambda \mapsto \phi(\lambda)\mathrm{Id}_{\mathcal{H}_0}$ and $\Psi \equiv \mathrm{Id}_{\mathcal{H}_0}$.

2) It is easy to show that for all $\Phi, \Psi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$, (Φ, Ψ) is ν-integrable and thus $\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}}$ is well defined as above.

3) In the special case where Φ and Ψ are valued in $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, $\mathcal{O}$-measurability reduces to simple measurability, (i) and (ii) are always verified, (iii) is equivalent to $\Phi f \Psi^{\mathsf{H}} \in \mathcal{L}^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{G}_0), \|\nu\|_1)$, in which case we have

$$\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} = \int \Phi f \Psi^{\mathsf{H}} \, \mathrm{d}\|\nu\|_1 \,.$$

In particular, since $\left\|\Phi f \Phi^{\mathsf{H}}\right\|_1 \leq \|\Phi\|^2 \|f\|_1 = \|\Phi\|^2$, $\|\nu\|_1-$a.e., we get that

$$\mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1) \subset \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) \,.$$

Moreover, the mapping defined, for all $\Phi, \Psi \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$, by $[\Phi, \Psi]_\nu := \int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}}$ is a pseudo Gramian in the sense that it satisfies all assumptions of Definition 2.5 except assumption (ii). In particular, the space $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ is larger than the ones used in [22] and [27, Appendix B.2.3] for filtering functional time series.

The following theorem, which corresponds to [19, Theorem 4.19] and Theorem 11 in [15, Section 3.4], shows that the same Gramian can be used over the larger space $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ and that it makes this space a normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module when quotiented by the set with zero norm.

Theorem 3.1. *Let $\mathcal{H}_0, \mathcal{G}_0$ be separable Hilbert spaces, $(\Lambda, \mathcal{A})$ a measurable space, ν a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ and $f = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$. Then $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ is an $\mathcal{L}_b(\mathcal{G}_0)$-module with module action*

$$\mathrm{Q} \bullet \Phi : \lambda \mapsto \mathrm{Q}\Phi(\lambda), \quad \mathrm{Q} \in \mathcal{L}_b(\mathcal{G}_0), \Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) .$$

Moreover, we can endow $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ with the pseudo-Gramian

$$[\Phi, \Psi]_\nu := \int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} \quad \Phi, \Psi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) . \tag{3.4}$$

Then, for all $\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$, we have

$$\|\Phi\|_\nu = \left\|[\Phi, \Phi]_\nu\right\|_1^{1/2} = 0 \Longleftrightarrow \Phi f^{1/2} = 0 \quad \|\nu\|_1\text{-a.e.}$$

Let us denote the class of such Φ's by $\{\|\cdot\|_\nu = 0\}$ and the quotient space by

$$\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) := \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) \Big/ \{\|\cdot\|_\nu = 0\} .$$

Then $\left(\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu), [\cdot, \cdot]_\nu\right)$ is a normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module.

3.3 Integration with respect to a random g.o.s. measure

We now define the mapping which provides a representation of the normal Hilbert $\mathcal{L}_b$-module module generated by a random g.o.s. measure in the form a module of square integrable operator functions. It is often seen as a stochastic integral because it linearly and continuously maps a function to a random variable. Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces, $(\Lambda, \mathcal{A})$ be a measurable space, and let ν be a trace-class p.o.v.m. defined on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. Given an $\mathcal{H}_0$-valued random g.o.s. measure W, we further set

$$\mathcal{H}^{W,\mathcal{G}_0} := \overline{\mathrm{Span}}^{\mathcal{G}} \left(\mathrm{Q}W(A) \; : \; \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \; A \in \mathcal{A} \right) , \tag{3.5}$$

which is a submodule of $\mathcal{G} := \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$. As in Proposition 13 in [15, Secion 3.4] and [19, Theorem 6.9], we now define the integral of an $\mathcal{H}_0 \to \mathcal{G}_0$ operator-valued function with respect to a random g.o.s. measure W as a Gramian-isometry from the normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$ to $\mathcal{H}^{W,\mathcal{G}_0}$. A detailed proof can be found in Section 5.2.

Theorem 3.2. *Let $(\Lambda, \mathcal{A})$ be a measurable space and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces. Let W be an $\mathcal{H}_0$-valued random g.o.s. measure on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure ν_W. Let $\mathcal{H}^{W,\mathcal{G}_0}$ be defined as in (3.5). Then there exists a unique Gramian-isometry*

$$I_W^{\mathcal{G}_0} : \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W) \to \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$$

such that, for all $A \in \mathcal{A}$ and $\mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$,

$$I_W^{\mathcal{G}_0}(1_A \mathrm{Q}) = \mathrm{Q}W(A) \quad \mathbb{P}\text{-a.s.}$$

Moreover, $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$ and $\mathcal{H}^{W,\mathcal{G}_0}$ are Gramian-isometrically isomorphic.

We can now define the integral of an operator-valued function with respect to W.

Definition 3.5 (Integral with respect to a random g.o.s. measure)**.** *Under the assumptions of Theorem 3.2, we use an integral sign to denote $I_W^{\mathcal{G}_0}(\Phi)$ for $\Phi \in \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$. Namely, we write*

$$\int \Phi \, \mathrm{d}W = \int \Phi(\lambda) \, W(\mathrm{d}\lambda) := I_W^{\mathcal{G}_0}(\Phi) . \tag{3.6}$$

The following remark will be useful.

Remark 3.2. *In the setting of Definition 3.5, take $\Phi = \phi \, \mathrm{Id}_{\mathcal{H}_0}$ with $\phi : \Lambda \to \mathbb{C}$. Then, we have $\Phi \in \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0), \nu_W)$ if and only if $\phi \in L^2(\Lambda, \mathcal{A}, \|\nu_W\|_1)$. We will omit $\mathrm{Id}_{\mathcal{H}_0}$ in the notation of the integral, writing $\int \phi \, \mathrm{d}W$ for $\int \phi \mathrm{Id}_{\mathcal{H}_0} \, \mathrm{d}W$.*

4 Modular spectral domain of a weakly stationary process and applications

4.1 The Gramian-Cramér representation and general Bochner theorems

We now have all the tools to derive a spectral theory for Hilbert-valued weakly stationary processes following [15, Section 4.2]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{H}_0$ be a separable Hilbert space and $(\mathrm{G}, +)$ be a locally compact Abelian (l.c.a.) group (for example $\mathbb{Z}$ or $\mathbb{R}$), whose null element is denoted by 0. This means that G is an Abelian topological group which is locally compact, Hausdorff for its topology. Recall that the dual group $\hat{\mathrm{G}}$ denotes the set of continuous characters of G i.e. the set of continuous functions $\chi : \mathrm{G} \to \mathbb{U}$ satisfying $s, t \in \mathrm{G}$, $\chi(s+t) = \chi(s)\chi(t)$, where $\mathbb{U}$ denotes the complex unit circle. In particular, $\hat{\mathbb{Z}} = \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ and $\hat{\mathbb{R}} = \mathbb{R}$. Details about l.c.a. groups can be found in [25]. Throughout this section we are interested in the spectral properties of a centered process valued in a separable Hilbert space and assumed to be weakly stationary in the following sense.

Definition 4.1 (Hilbert-valued weakly stationary processes)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{H}_0$ be a separable Hilbert space and $(\mathrm{G}, +)$ be an l.c.a. group. Then a process $X := (X_t)_{t \in \mathrm{G}}$ is said to be an $\mathcal{H}_0$-valued weakly stationary process if*

(i) For all $t \in \mathrm{G}$, $X_t \in L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$.

(ii) For all $t \in \mathrm{G}$, $\mathbb{E}[X_t] = \mathbb{E}[X_0]$. We say that X is centered if $\mathbb{E}[X_0] = 0$.

(iii) For all $t, h \in \mathrm{G}$, $\mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}(X_h, X_0)$.

(iv) The autocovariance operator function *$\Gamma_X : h \mapsto \mathrm{Cov}(X_h, X_0)$ satisfies the following continuity condition: for all $\mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0)$, $h \mapsto \mathrm{Tr}(\mathrm{Q}\Gamma_X(h))$ is continuous on G.*

In the case of time series, $\mathrm{G} = \mathbb{Z}$, all mappings on G are continuous and Condition (iv) can be discarded in this definition. It is less trivial to show that, for any l.c.a. group G, we get an equivalent definition if we replace (iv) by just saying that Γ_X is continuous in w.o.t. This interesting fact is explained in the following remark in a more detailed fashion.

Remark 4.1. *For any $x, y \in \mathcal{H}_0$, taking $\mathrm{Q} = xy^{\mathsf{H}}$ we have $\mathrm{Tr}(\mathrm{Q}\Gamma_X(h)) = \langle \Gamma_X(h)x, y \rangle_{\mathcal{H}_0}$. Hence Condition (iv) of Definition 4.1 implies the following one.*

(iv') The autocovariance operator function $\Gamma_X : h \mapsto \mathrm{Cov}(X_h, X_0)$ is continuous in w.o.t.

It is easy to find $\mathrm{G}, \mathcal{H}_0$ and a mapping $f : \mathrm{G} \to \mathcal{S}_1(\mathcal{H}_0)$ which is continuous in w.o.t. but such that $h \mapsto \mathrm{Tr}(f(h))$ is not continuous hence does not satisfy the continuity condition imposed on Γ_X in (iv). However, it turns out that if Γ_X is the autocovariance operator function $h \mapsto \mathrm{Cov}(X_h, X_0)$ with X satisfying Conditions (i) and (iii), then Conditions (iv) and (iv') become equivalent. The reason behind this surprising fact will be made clear later in Point 1) of Remark 4.2. In other words, we can replace (iv) by (iv') without altering Definition 4.1.

As in the univariate case, the notion of weak stationarity is related to an isometric property of the lag operators, but here the covariance stationarity expressed in Condition (iii) translates into a Gramian-isometric property rather than a scalar isometric property. Namely, let $X := (X_t)_{t \in \mathrm{G}}$ satisfy Conditions (i) and (ii) and take it centered so that each X_t belongs to the normal Hilbert module $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ as defined in Example 2.1. For all $h \in \mathrm{G}$, define the lag operator of lag $h \in \mathrm{G}$ as the mapping $U_h^X : X_t \mapsto X_{t+h}$ defined for all $t \in \mathrm{G}$. Then Condition (iii) is equivalent to saying that for all $h \in \mathrm{G}$, the mapping U_h^X is Gramian-isometric on $\{X_t : t \in \mathrm{G}\}$ for the Gramian structure inherited from $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$. Thus, if this condition holds, by Proposition 2.2, for any lag $h \in \mathrm{G}$, there exists a unique Gramian-unitary operator extending U_h^X on the *modular time domain* $\mathcal{H}^X$ of X defined as the submodule of $\mathcal{H}$ generated by the X_t's, that is,

$$\mathcal{H}^X := \overline{\mathrm{Span}}^{\mathcal{H}} \left(\mathrm{Q}X_t \; : \; \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0), \, t \in \mathrm{G} \right) ,$$

which is the generalization of (1.5) to a general l.c.a. group G. In fact it is convenient to introduce a slightly more general definition of the modular time domain where the output Hilbert space $\mathcal{G}_0$ may be taken different form $\mathcal{H}_0$.

Definition 4.2 ($\mathcal{G}_0$-valued modular time domain). *Let* $(\mathrm{G},+)$ *be an l.c.a. group, and* $\mathcal{H}_0$ *and* $\mathcal{G}_0$ *be two separable Hilbert spaces. Let* $X := (X_t)_{t\in\mathrm{G}}$ *be a collection of variables in* $\mathcal{M}(\Omega,\mathcal{F},\mathcal{H}_0,\mathbb{P})$ *as defined in Example 2.1. The* $\mathcal{G}_0$*-valued* modular time domain *of* X *is defined by*

$$\mathcal{H}^{X,\mathcal{G}_0} := \overline{\mathrm{Span}}^{\mathcal{M}(\Omega,\mathcal{F},\mathcal{G}_0,\mathbb{P})}\left(\mathrm{Q}X_t \;:\; \mathrm{Q}\in\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\, t\in\mathrm{G}\right) , \tag{4.1}$$

which is a submodule of $\mathcal{M}(\Omega,\mathcal{F},\mathcal{G}_0,\mathbb{P})$.

We now extend the (scalar) Cramér representation theorem by means of an integral with respect to a g.o.s. measure.

Theorem 4.1 (Gramian-Cramér representation theorem). *Let* $\mathcal{H}_0$ *be a separable Hilbert space,* $(\Omega,\mathcal{F},\mathbb{P})$ *be a probability space and* $(\mathrm{G},+)$ *be an l.c.a. group. Let* $X := (X_t)_{t\in\mathrm{G}}$ *be a centered weakly stationary* $\mathcal{H}_0$*-valued process as in Definition 4.1. Then there exists a unique regular* $\mathcal{H}_0$*-valued random g.o.s. measure* $\hat{X}$ *on* $(\hat{\mathrm{G}},\mathcal{B}(\hat{\mathrm{G}}),\Omega,\mathcal{F},\mathbb{P})$ *such that*

$$X_t = \int \chi(t)\;\hat{X}(\mathrm{d}\chi) \quad \textit{for all} \quad t\in\mathrm{G}\,. \tag{4.2}$$

This result is partly stated in Theorem 2 in [15, Section 4.2]. Here we add the uniqueness of $\hat{X}$, which appears to be a new result in this general setting. We provide a detailed proof in Section 5.3. In fact Theorem 2 in [15, Section 4.2] contains a converse statement, which we now state separately as a lemma whose detailed proof can be found in Section 5.3.

Lemma 4.2. *Let* $(\mathrm{G},+)$ *be an l.c.a. group,* $\mathcal{H}_0$ *a separable Hilbert space and* W *be an* $\mathcal{H}_0$*-valued random g.o.s. measure on* $(\hat{\mathrm{G}},\mathcal{B}(\hat{\mathrm{G}}),\Omega,\mathcal{F},\mathbb{P})$ *with intensity operator measure* ν*. Define, for all* $t\in\mathrm{G}$,

$$X_t = \int \chi(t)\;W(\mathrm{d}\chi)\,.$$

Then $X = (X_t)_{t\in\mathrm{G}}$ *is a centered* $\mathcal{H}_0$*-valued weakly stationary process with autocovariance operator function* Γ *defined by*

$$\Gamma(h) = \int \chi(h)\,\nu(\mathrm{d}\chi) \quad \textit{for all } h\in\mathrm{G}. \tag{4.3}$$

With Theorem 4.1 at our disposal, we can now define the *Gramian-Cramér representation* and the *spectral operator measure* of X.

Definition 4.3 (Gramian-Cramér representation and spectral operator measure). *Under the setting of Theorem 4.1, the regular g.o.s. measure* $\hat{X}$ *is called the* (Gramian) Cramér representation *of* X *and its intensity operator measure is called the* spectral operator measure *of* X*. It is a regular trace-class p.o.v.m. on* $(\hat{\mathrm{G}},\mathcal{B}(\hat{\mathrm{G}}),\mathcal{H}_0)$.

By Lemma 4.2, we see that the autocovariance operator function and the spectral operator measure of X are related to each other through the identity (4.3). As already hinted in the introduction, using the tools introduced in Section 3.3, we can more generally interpret the Cramér representation of Theorem 4.1 as establishing a Gramian-isometric mapping onto the modular time domain of X, starting from its *modular spectral domain* which we now introduce.

Definition 4.4 ($\mathcal{G}_0$-valued spectral time domain). *Let* $\mathcal{H}_0$ *and* $\mathcal{G}_0$ *be two separable Hilbert spaces and* $X := (X_t)_{t\in\mathrm{G}}$ *be a centered weakly stationary process valued in* $\mathcal{H}_0$ *as in Definition 4.1. The* $\mathcal{G}_0$*-valued* modular spectral domain *of* X *is the normal Hilbert* $\mathcal{L}_b(\mathcal{G}_0)$*-module defined by*

$$\widehat{\mathcal{H}}^{X,\mathcal{G}_0} := \mathsf{L}^2(\hat{\mathrm{G}},\mathcal{B}(\hat{\mathrm{G}}),\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu_X)\,, \tag{4.4}$$

where ν_X *is the spectral operator measure of* X *introduced in Definition 4.3.*

We can now state that the modular time and spectral domain are Gramian-isometrically isomorphic, whose proof can be found in Section 5.3.

Theorem 4.3 (Kolmogorov isomorphism theorem). *Under the setting of Theorem 4.1, for any separable Hilbert space* $\mathcal{G}_0$*, the mapping* $I_{\hat{X}}^{\mathcal{G}_0} : \Phi \mapsto \int \Phi\;\mathrm{d}\hat{X}$ *is a Gramian-unitary operator from* $\widehat{\mathcal{H}}^{X,\mathcal{G}_0}$ *to* $\mathcal{H}^{X,\mathcal{G}_0}$ *and we have* $\mathcal{H}^{X,\mathcal{G}_0} = \mathcal{H}^{\hat{X},\mathcal{G}_0}$*. Thus, the* $\mathcal{G}_0$*-valued* modular time domain $\mathcal{H}^{X,\mathcal{G}_0}$ *and the* $\mathcal{G}_0$*-valued* modular spectral domain $\widehat{\mathcal{H}}^{X,\mathcal{G}_0}$ *are* Gramian-isometrically isomorphic.

Relation (4.3) is at the core of the general Bochner theorem, which we now discuss. Recall that the *standard* (univariate) Bochner theorem can be stated as follows (see [25, Theorem 1.4.3] for existence and [25, Theorem 1.3.6] for uniqueness).

Theorem 4.4 (Bochner Theorem)**.** *Let* $(\mathrm{G},+)$ *be an l.c.a. group and* $\gamma : \mathrm{G} \to \mathbb{C}$*. Then the two following statements are equivalent:*

(i) γ *is continuous and hermitian non-negative definite, that is, for all* $n \in \mathbb{N}$*,* $t_1, \cdots, t_n \in \mathrm{G}$ *and* $a_1, \cdots, a_n \in \mathbb{C}$*,*

$$\sum_{i,j=1}^{n} a_i \overline{a_j} \gamma(t_i - t_j) \geq 0.$$

(ii) There exists a regular finite non-negative measure ν *on* $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}))$ *such that*

$$\gamma(h) = \int \chi(h)\, \nu(\mathrm{d}\chi), \quad h \in \mathrm{G}. \tag{4.5}$$

Moreover, if Assertion (ii) holds, ν *is the unique regular non-negative measure satisfying* (4.5).

There are various other ways to extend Condition (i) of Theorem 4.4 when replacing $\mathbb{C}$ by a Hilbert space $\mathcal{H}_0$.

Definition 4.5. *Let* $\mathcal{H}_0$ *be a Hilbert space and* $(\mathrm{G},+)$ *an l.c.a. group. A function* $\Gamma : \mathrm{G} \to \mathcal{L}_b(\mathcal{H}_0)$ *is said to be*

1. *a* proper autocovariance *operator function if* $\mathcal{H}_0$ *is separable and there exists a* $\mathcal{H}_0$*-valued weakly stationary process with autocovariance operator function* Γ*;*
2. positive definite *if for all* $n \in \mathbb{N}^*$*,* $t_1, \cdots, t_n \in \mathrm{G}$ *and* $\mathrm{Q}_1, \cdots, \mathrm{Q}_n \in \mathcal{L}_b(\mathcal{H}_0)$*,*

$$\sum_{i,j=1}^{n} \mathrm{Q}_i \Gamma(t_i - t_j) \mathrm{Q}_j^{\mathsf{H}} \succeq 0\ ;$$

3. *of* positive-type *if for all* $n \in \mathbb{N}^*$*,* $t_1, \cdots, t_n \in \mathrm{G}$ *and* $x_1, \cdots, x_n \in \mathcal{H}_0$*,*

$$\sum_{i,j=1}^{n} \langle \Gamma(t_i - t_j) x_j, x_i \rangle_{\mathcal{H}_0} \geq 0\ ;$$

4. hermitian non-negative definite *if for all* $n \in \mathbb{N}^*$*,* $t_1, \cdots, t_n \in \mathrm{G}$ *and* $a_1, \cdots, a_n \in \mathbb{C}$*,*

$$\sum_{i,j=1}^{n} a_i \overline{a_j} \Gamma(t_i - t_j) \succeq 0.$$

Equivalently, Γ *is hermitian non-negative definite if and only if for all* $x \in \mathcal{H}_0$*,* $t \mapsto \langle \Gamma(t)x, x \rangle_{\mathcal{H}_0}$ *is hermitian non-negative definite.*

It is straightforward to show that the definitions in Definition 4.5 are given in an increasing order of generality in the sense that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$. In the univariate case, for a continuous $\gamma : \mathrm{G} \to \mathbb{C}$ all these definitions are trivially equivalent to Assertion (i) in Theorem 4.4. A natural question for a general Hilbert space $\mathcal{H}_0$ is which definition should be used to extend the Bochner theorem. A first answer is the following corollary whose proof can be found in Section 5.3

Corollary 4.5. *Let* $(\mathrm{G},+)$ *be an l.c.a. group,* $\mathcal{H}_0$ *a separable Hilbert space and* $\Gamma : \mathrm{G} \to \mathcal{L}_b(\mathcal{H}_0)$*. Then the following assertions are equivalent.*

(i) The function Γ *is a proper autocovariance operator function.*

(ii) There exists a regular trace-class p.o.v.m. ν *on* $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{H}_0)$ *such that (4.3) holds.*

This result extends Bochner's theorem from the point of view of $\mathcal{H}_0$-valued weakly stationary processes so that Γ in Corollary 4.5(i) is valued in $\mathcal{S}_1(\mathcal{H}_0)$ and for all $\mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0)$, $h \mapsto \mathrm{Tr}(\mathrm{Q}\Gamma(h))$ is continuous. It turns our that other extensions can be obtained using a purely operator theory point of view with the more general positiveness conditions of Definition 4.5. In the following theorem, $\mathcal{H}$ is not necessarily separable, Γ is not necessarily $\mathcal{S}_1(\mathcal{H})$-valued (and therefore the resulting p.o.v.m. may not be trace-class) and its continuity condition can be relaxed to continuity for the w.o.t. This result is essentially the Naimark's moment theorem of [3]. We refer to it as the *general* Bochner theorem (or general Herglotz theorem for $\mathrm{G} = \mathbb{Z}$).

Theorem 4.6 (General Bochner Theorem)**.** *Let* $(\mathrm{G},+)$ *be an l.c.a. group,* $\mathcal{H}$ *a Hilbert space and* $\Gamma : \mathrm{G} \to \mathcal{L}_b(\mathcal{H})$*. Then the following assertions are equivalent.*

(i) Γ *is continuous in w.o.t. and positive definite.*

(ii) Γ *is continuous in w.o.t. and of positive type.*

(iii) Γ *is continuous in w.o.t. and hermitian non-negative definite.*

(iv) There exists a regular p.o.v.m. ν *on* $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{H})$ *such that (4.3) holds.*

Moreover, if Assertion (iv) holds, ν *is the unique regular p.o.v.m. satisfying (4.3).*

It is important to note that there is a subtle difference between Assertion (ii) of Corollary 4.5 and Assertion (iv) of Theorem 4.6, namely, the latter assertion is weaker since ν is not supposed to be trace-class. In particular, we cannot rely on the Radon-Nikodym derivative as ν is not trace-class. The proof of Theorem 4.6 is discussed in Section 4.1. An immediate consequence of Corollary 4.5 and Theorem 4.6 is the following result whose proof can be found in Section 4.1.

Corollary 4.7. *Let* $(\mathrm{G},+)$ *be an l.c.a. group,* $\mathcal{H}_0$ *a separable Hilbert space and* $\Gamma : \mathrm{G} \to \mathcal{L}_b(\mathcal{H}_0)$*. Then the following assertions are equivalent.*

(i) The function Γ *is a proper autocovariance operator function.*

(ii) Any of the Assertions (i)–(iii) in Theorem 4.6 holds and $\Gamma(0) \in \mathcal{S}_1(\mathcal{H}_0)$*.*

Remark 4.2. *Let us briefly comment on the equivalence established in Corollary 4.7.*

1) *In Condition (iv) of Definition 4.1, we required a condition on* Γ *which is stronger than continuity in w.o.t. However in Assertion (ii) of Corollary 4.7, the continuity of* Γ *is only needed in the w.o.t. This means that we can replace the continuity Condition (iv) in Definition 4.1 by continuity in w.o.t. as in Remark 4.1 (iv') without changing the overall definition of a weakly stationary process.*

2) *The previous remark is related to a fact established in Proposition 3 in [15, Section 4.2], which states the equivalence between being* scalar stationary *and being* operator stationary*. The latter definition is the same as our Definition 4.1, and the former one amounts to replace Condition (iv) in Definition 4.1 by assuming that for all* $x \in \mathcal{H}_0$*,* $x^{\mathsf{H}}\Gamma x : h \mapsto x^{\mathsf{H}}\Gamma(h)x$ *is continuous and hermitian non-negative definite. But this amounts to says that* Γ *itself is continuous in the w.o.t. and hermitian non-negative definite. Since* $\Gamma(0) \in \mathcal{S}_1(\mathcal{H}_0)$ *is a consequence of Assertion (i) in Definition 4.1, Corollary 4.7 indeed implies the equivalence established by Proposition 3 in [15, Section 4.2].*

4.2 Composition and inversion of filters

With the construction of the spectral theory for weakly stationary processes of Section 4.1, the study of linear filters for such processes is easily derived. Indeed, we are now able to give the most general definition of linear filtering, characterize the spectral structure of the filtered process and provide results on compositions and inversion of linear filters. Then, in the next section, we will provide a general statement of harmonic principal component analysis for weakly stationary processes valued in a separable Hilbert space.

Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces and $\Phi \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$. Let $X = (X_t)_{t \in \mathrm{G}}$ be an $\mathcal{H}_0$-valued weakly stationary stochastic process such that

$$\Phi \in \widehat{\mathcal{H}}^{X, \mathcal{G}_0} , \tag{4.6}$$

where $\widehat{\mathcal{H}}^{X, \mathcal{G}_0}$ denotes the modular spectral domain of Definition 4.4. Then we can define a $\mathcal{G}_0$-valued weakly stationary stochastic process $Y = (Y_t)_{t \in \mathrm{G}}$ by

$$Y_t = \int \chi(t)\, \Phi(\chi)\, \hat{X}(\mathrm{d}\chi) , \quad \text{for all } t \in \mathrm{G} . \tag{4.7}$$

We say that Φ is the *transfer operator function* of the filter. For convenience we write, in the time domain,

$$X \in \mathcal{S}_{\Phi}(\Omega, \mathcal{F}, \mathbb{P}) \quad \text{and} \quad Y = F_{\Phi}(X) , \tag{4.8}$$

for (4.6) and (4.7) respectively. Many examples in the literature rely on a *time domain* description of the filtering obtained as in the following example.

Example 4.1 (Convolutional filtering in the discrete case)**.** *Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces. Let $X = (X_t)_{t\in\mathbb{Z}}$ be an $\mathcal{H}_0$-valued weakly stationary stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\Phi = (\Phi_k)_{k\in\mathbb{Z}}$ be a sequence valued in $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ such that $\sum_k \|\Phi_k\| < \infty$. Define the process $Y = (Y_t)_{t\in\mathrm{G}}$ by the* time domain convolutional filtering

$$Y_t = \sum_{k\in\mathbb{Z}} \Phi_k\, X_{t-k}\ , \quad t \in \mathrm{G}\ ,$$

which converges absolutely in $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$. Then, defining $\hat{\Phi} : \mathbb{T} \to \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ by

$$\hat{\Phi}(\lambda) = \sum_{k\in\mathbb{Z}} \Phi_k\, \mathrm{e}^{\mathrm{i}\lambda k}\ ,$$

which absolutely converges in $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, we can show that $\hat{\Phi} \in \mathsf{L}^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_X)$ and $Y = F_{\hat{\Phi}}(X)$.

This example can be easily extended to processes indexed by any l.c.a. group G by using the Fubini-type theorem Proposition 5.6, see Example 5.1.

The following result deals with the composition and inversion of general filters. Its proof can be found in Section 5.4. See also Section 2.3, where we introduced the symbols $\cong$, $\lesssim$.

Proposition 4.8 (Composition and inversion of filters on weakly stationary time series)**.** *Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces and pick a transfer operator function $\Phi \in \mathbb{F}_{\mathcal{O}}\left(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{H}_0, \mathcal{G}_0\right)$. Let X be a centered weakly stationary $\mathcal{H}_0$-valued process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with spectral operator measure ν_X. Suppose that $X \in \mathcal{S}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$ and set $Y = F_\Phi(X)$, as defined in (4.8). Then the three following assertions hold.*

(i) For any separable Hilbert space $\mathcal{I}_0$, we have $\mathcal{H}^{Y,\mathcal{I}_0} \lesssim \mathcal{H}^{X,\mathcal{I}_0}$.

(ii) For any separable Hilbert space $\mathcal{I}_0$ and all $\Psi \in \mathbb{F}_{\mathcal{O}}\left(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{G}_0, \mathcal{I}_0\right)$, we have $X \in \mathcal{S}_{\Psi\Phi}(\Omega, \mathcal{F}, \mathbb{P})$ if and only if $F_\Phi(X) \in \mathcal{S}_\Psi(\Omega, \mathcal{F}, \mathbb{P})$, and in this case, we have

$$F_\Psi \circ F_\Phi(X) = F_{\Psi\Phi}(X). \tag{4.9}$$

(iii) Suppose that Φ is injective $\|\nu_X\|_1$-a.e. Then $X = F_{\Phi^{-1}} \circ F_\Phi(X)$, where we define $\Phi^{-1}(\lambda) := \left(\Phi(\lambda)_{|\mathcal{D}(\Phi(\lambda))\to\mathrm{Im}(\Phi(\lambda))}\right)^{-1}$ with domain $\mathrm{Im}(\Phi(\lambda))$ for all $\lambda \in \{\Phi$ is injective$\}$ and $\Phi^{-1}(\lambda) = 0$ otherwise. Moreover, Assertion (i) above holds with $\lesssim$ replaced by $\cong$.

4.3 Cramér-Karhunen-Loève decomposition

Let $\mathcal{H}_0$ be a separable Hilbert space with (possibly infinite) dimension N and $X = (X_t)_{t\in\mathrm{G}}$ be a centered, $\mathcal{H}_0$-valued weakly-stationary process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Cramér representation $\hat{X}$ and spectral operator measure ν_X.

The Cramér-Karhunen-Loève decomposition amounts to give a rigorous meaning to the formula

$$\hat{X}(\mathrm{d}\chi) = \sum_{0\leq n<N} \phi_n(\chi) \otimes \phi_n(\chi)\, \hat{X}(\mathrm{d}\chi)\ , \tag{4.10}$$

where, for all $\chi \in \hat{\mathrm{G}}$, $(\phi_n(\chi))_{0\leq n<N}$ is an orthonormal sequence in $\mathcal{H}_0$ chosen in such a way that the summands in (4.10) are uncorrelated and where, for all $x \in \mathcal{H}$ and $y \in \mathcal{G}$, we denote by $x \otimes y$ the trace-class operator from $\mathcal{G}$ onto $\mathcal{H}$ defined by $(x \otimes y)z = \langle z, y\rangle_{\mathcal{G}}\, x$ for all $z \in \mathcal{G}$. Such a decomposition provides a way to derive the harmonic principal component analysis of the process X, which is an approximation of X by a finite rank linear filtering. In recent works, the functional Cramér-Karhunen-Loève decomposition is achieved under additional assumptions on ν_X such as having a continuous density with respect to the Lebesgue measure (in [26]) or at most finitely many atoms (in [28]). In fact, thanks to the Radon-Nikodym property of trace-class p.o.v.m.'s, there is no need for such additional assumptions. Instead, we rely on the following lemma, whose proof can be found in Section 5.5.

Lemma 4.9 (Eigendecomposition of a trace-class p.o.v.m.)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space with dimension $N \in \{1, \ldots, +\infty\}$. Let ν be a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ and μ a σ-finite dominating measure of ν, e.g. its variation norm $\|\nu\|_1$. Then there exist sequences $(\sigma_n)_{0\leq n<N}$ and $(\phi_n)_{0\leq n<N}$ of $(\Lambda, \mathcal{A}) \to (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ and $(\Lambda, \mathcal{A}) \to (\mathcal{H}_0, \mathcal{B}(\mathcal{H}_0))$ measurable functions, respectively, such that the following assertions hold.*

(i) *For all* $\lambda \in \Lambda$, $(\sigma_n(\lambda))_{0 \le n < N}$ *is non-increasing and* $\sum_{0 \le n < N} \sigma_n(\lambda) < \infty$.

(ii) *For all* $\lambda \in \Lambda$, $(\phi_n(\lambda))_{0 \le n < N}$ *is orthonormal.*

(iii) *The trace-class p.o.v.m.* ν *admits the density*

$$f : \lambda \mapsto \sum_{0 \le n < N} \sigma_n(\lambda)\, \phi_n(\lambda) \otimes \phi_n(\lambda) \ ,$$

with respect to μ*, where the convergence holds absolutely in* $\mathcal{S}_1$ *for each* $\lambda \in \Lambda$.

Moreover, using the notations $\phi_n^{\mathsf{H}} : \lambda \mapsto \phi_n(\lambda)^{\mathsf{H}}$ *and* $\phi_n \otimes \phi_n : \lambda \mapsto \phi_n(\lambda) \otimes \phi_n(\lambda)$*, we have the following properties.*

(iv) *The sequence* $(\phi_n^{\mathsf{H}})_{0 \le n < N}$ *is orthogonal in* $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathbb{C}), \nu)$.

(v) *The sequence* $(\phi_n \otimes \phi_n)_{0 \le n < N}$ *is Gramian-orthogonal in* $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0), \nu)$.

(vi) *The* $\mathcal{L}_b(\mathcal{H}_0)$*-valued mapping* $\sum_{0 \le n < N} \phi_n \otimes \phi_n$ *is equal to the mapping* $\lambda \mapsto \mathrm{Id}_{\mathcal{H}_0}$ *in* $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0), \nu)$.

Assertion (vi) may be misleading at first sight, so the following comment may be worth noting.

Remark 4.3. *By Assertions (i)-(iii), for all* $\lambda \in \Lambda$, $\sum_{0 \le n < N} \phi_n(\lambda) \otimes \phi_n(\lambda)$ *is the orthogonal projection onto the closure of the range of* $f(\lambda)$*. Thus, Assertion (vi) says that this projection is equal to* $\mathrm{Id}_{\mathcal{H}_0}$ *in* $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0), \nu)$*. It is not equivalent to saying that* $\sum_{0 \le n < N} \phi_n \otimes \phi_n = \mathrm{Id}_{\mathcal{H}_0}$, $\|\nu\|_1$*-a.e. since it may happen that the range of* $f(\lambda)$ *is dense in* $\mathcal{H}_0$ *for none of the* λ*'s, in which case we have Assertion (vi) at the same time as* $\{\sum_{0 \le n < N} \phi_n \otimes \phi_n = \mathrm{Id}_{\mathcal{H}_0}\} = \emptyset$.

We then get the following general formulation of a harmonic principal components analysis for $\mathcal{H}_0$-valued weakly-stationary processes, whose proof can be found in Section 5.5. We also refer to Remark 5.1 in the same section, where we explain how to correctly interpret (4.10).

Proposition 4.10 (Harmonic functional principal components analysis)**.** *Let* $\mathcal{H}_0$ *be a separable Hilbert space and* $X = (X_t)_{t \in \mathrm{G}}$ *be a centered,* $\mathcal{H}_0$*-valued weakly-stationary process defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with spectral operator measure* ν_X*. Let* $(\sigma_n)_{0 \le n < N}$ *and* $(\phi_n)_{0 \le n < N}$ *be given as in Lemma 4.9 for some dominating measure* μ *of* ν_X*, for instance* $\mu = \|\nu_X\|_1$*. Let* $q : \hat{\mathrm{G}} \to \mathbb{N}^*$ *be a measurable function. Then for all* $t \in \mathrm{G}$,

$$\min \left\{ \mathbb{E}\left[\left\| X_t - [F_\Theta(X)]_t \right\|_{\mathcal{H}_0}^2 \right] \ : \ \Theta \in \mathsf{L}^2(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{O}(\mathcal{H}_0), \nu_X),\ \mathrm{rank}(\Theta) \le q \right\}$$

is equal to

$$\int_{\hat{\mathrm{G}}} \sum_{q(\chi) \wedge N \le n < N} \sigma_n(\chi)\, \mu(\mathrm{d}\chi) \ ,$$

and the minimum is achieved for

$$\Theta : \chi \mapsto \sum_{0 \le n < q(\chi) \wedge N} \phi_n(\chi) \otimes \phi_n(\chi) \ .$$

5 Postponed proofs

5.1 Proofs of Section 2

We start with a useful lemma about measurability of compact and Schatten operator-valued functions.

Lemma 5.1. *Let* $(\Lambda, \mathcal{A})$ *be a measurable space and* $\mathcal{H}_0, \mathcal{G}_0$ *two separable Hilbert spaces. Let* $\mathcal{E} = \mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$ *or* $\mathcal{S}_p(\mathcal{H}_0, \mathcal{G}_0)$ *where* $p \in \{1, 2\}$ *and. Then a function* $\Phi : \Lambda \to \mathcal{E}$ *is measurable if and only if it is simply measurable.*

Proof. By (2.1), we only need to show that, if Φ is simply measurable then it is measurable. Since the space $\mathcal{E}$ is separable, Pettis's measurability theorem gives that it is enough to show that for all $f \in \mathcal{E}^*$, $f \circ \Phi$ is a measurable complex-valued function. By [8, Theorems 19.1, 18.14, 19.2], we get that $\mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)^*$, $\mathcal{S}_1(\mathcal{H}_0, \mathcal{G}_0)^*$ and $\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)^*$ are respectively isometrically isomorphic to $\mathcal{S}_1(\mathcal{H}_0, \mathcal{G}_0)$, $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and $\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$ and the duality relation can be defined on $\mathcal{E} \times \mathcal{E}^*$ as $(\mathrm{Q}, \mathrm{T}) \mapsto \mathrm{Tr}(\mathrm{T}^{\mathsf{H}}\mathrm{Q})$. This means that we only have to show measurability of the complex-valued functions $\lambda \mapsto \mathrm{Tr}(\mathrm{Q}^{\mathsf{H}}\Phi(\lambda))$ for all $\mathrm{Q} \in \mathcal{E}^*$. Let $(\phi_k)_{k\in\mathbb{N}}, (\psi_k)_{k\in\mathbb{N}}$ be Hilbert basis of $\mathcal{H}_0$ and $\mathcal{G}_0$ respectively, then $\mathrm{Tr}(\mathrm{Q}^{\mathsf{H}}\Phi(\lambda)) = \sum_{k\in\mathbb{N}} \langle \Phi(\lambda)\phi_k, \mathrm{Q}\psi_k \rangle_{\mathcal{G}_0}$ which defines a measurable function of λ by simple measurability of Φ. □

We now provide the proof of Lemma 2.1

Proof of Lemma 2.1. The first point comes from the fact that for all $A \in \mathcal{A}$, $\nu(\Lambda) - \nu(A)$ is a positive operator. Now, if ν is trace-class, then (2.3) is easily verified for the norm $\|\cdot\|_1$ using the fact that $\|\cdot\|_1 = \mathrm{Tr}(\cdot)$ for positive operators. Finally, by definition of $\|\nu\|_1$, regularity of $\|\nu\|_1$ is equivalent to regularity of ν as an $\mathcal{S}_1(\mathcal{H}_0)$-valued measure which clearly implies regularity of $\nu_x = x^{\mathsf{H}}\nu(\cdot)x$ for all $x \in \mathcal{H}_0$. Suppose now that for all $x \in \mathcal{H}_0$, ν_x is regular, then let $(e_k)_{k\in\mathbb{N}}$ be a Hilbert basis of $\mathcal{H}_0$, and define for all $n \in \mathbb{N}$, the non-negative measure $\mu_n := \sum_{k=0}^{n} \nu_{e_k}$ such that for all $A \in \mathcal{A}$, $\|\nu\|_1(A) = \lim_{n\to+\infty} \mu_n(A) = \sup_{n\in\mathbb{N}} \mu_n(A)$. Then, by Vitali-Hahn-Sakh-Nikodym's theorem (see [4]), the sequence $(\mu_n)_{n\in\mathbb{N}}$ is uniformly countably additive which implies regularity of $\|\nu\|_1$ by Lemma 23 in [9, Chapter VI, Section 2]. □

We also provide the following useful properties on the density of a trace-class p.o.v.m. with respect to a dominating measure.

Lemma 5.2. *Les ν be a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$ and μ a σ-finite measure such that $\|\nu\|_1 \ll \mu$. Les $g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. Then the following assertions hold.*

- *(a) For μ-almost every $\lambda \in \Lambda$, $g(\lambda) \in \mathcal{S}_1^+(\mathcal{H}_0)$.*
- *(b) The mapping $g^{1/2} : \lambda \mapsto g(\lambda)^{1/2}$ belongs to $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0), \mu)$.*
- *(c) The density of $\|\nu\|_1$ with respect to μ is $\|g\|_1$. In particular, $g = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}\|g\|_1$ μ-a.e. and if $\mu = \|\nu\|_1$, then $\|g\|_1 = 1$ μ-a.e.*
- *(d) Let $f : \Lambda \to \mathbb{C}$ be measurable. Then $f \in L^1(\Lambda, \mathcal{A}, \|\nu\|_1)$ if and only if $\lambda \mapsto f(\lambda)\, g(\lambda) \in L^1(\Lambda, \mathcal{A}, \mathcal{S}_1(\mathcal{H}_0), \mu)$, and we have $\displaystyle\int f(\lambda)\, \nu(\mathrm{d}\lambda) = \int f(\lambda)\, g(\lambda)\, \mu(\mathrm{d}\lambda)$.*

Proof. For all $x \in \mathcal{H}_0$ and $A \in \mathcal{A}$,

$$\int_A \langle g(\lambda)x, x\rangle_{\mathcal{H}_0}\ \mu(\mathrm{d}\lambda) = \langle \nu(A)x, x\rangle_{\mathcal{H}_0} \geq 0\ ,$$

and there exists a set $A_x \in \mathcal{A}$ with $\mu(A_x^c) = 0$ and $\langle g(\lambda)x, x\rangle_{\mathcal{H}_0} \geq 0$ for all $\lambda \in A_x$. Taking $(x_n)_{n\in\mathbb{N}}$ a dense countable subset of $\mathcal{H}_0$ we get that $g \in \mathcal{S}_1^+(\mathcal{H}_0)$ on $A = \bigcap_{n\in\mathbb{N}} A_{x_n}$ thus proving Assertion (a). For Assertion (b), we get have $g^{1/2} \in \mathbb{F}(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0))$ by Lemma 2 in [15, Section 3.4] and Lemma 5.1 and $g^{1/2} \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0), \mu)$ then follows from the identity $\left\|g^{1/2}(\lambda)\right\|_2^2 = \|g(\lambda)\|_1$. Moreover, taking the trace in (2.4) gives for all $A \in \mathcal{A}$,

$$\|\nu\|_1(A) = \int_A \|g\|_1 \,\mathrm{d}\mu$$

which gives Assertion (c). Finally, Assertion (d) is easy to get by extending the case $f = \mathbb{1}_A$ for $A \in \mathcal{A}$ to simple functions and then using the density of simple functions. □

5.2 Proofs of Section 3

We start by exhibiting the relation between the spaces $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ and $L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$ where $\|\nu\|_1 \ll \mu$. It is easy to show that this last space is a normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module with module action defined, for $\mathrm{Q} \in \mathcal{L}_b(\mathcal{G}_0)$ and $\Phi \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$, by $\mathrm{Q} \bullet \Phi : \lambda \mapsto \mathrm{Q}\Phi(\lambda)$ and Gramian defined, for $\Phi, \Psi \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$, by

$$[\Phi, \Psi]_{L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)} := \int \Phi\Psi^{\mathsf{H}}\ \mathrm{d}\mu\ . \tag{5.1}$$

The following proposition provides an easy way to verify that a function belongs in $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$.

Proposition 5.3. *Let $(\Lambda,\mathcal{A})$ be a measurable space, $\mathcal{H}_0,\mathcal{G}_0,\mathcal{I}_0$ be three separable Hilbert spaces and ν a trace-class p.o.v.m. on $(\Lambda,\mathcal{A},\mathcal{H}_0)$. Let μ be a σ-finite non-negative measure dominating $\|\nu\|_1$ and set $g=\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. Then the following assertions hold.*

(a) For all $\Phi\in\mathbb{F}_{\mathcal{O}}\left(\Lambda,\mathcal{A},\mathcal{H}_0,\mathcal{G}_0\right)$, we have $\Phi\in\mathscr{L}^2(\Lambda,\mathcal{A},\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu)$ if and only if

$$\begin{cases}\mathrm{Im}(g^{1/2})\subset\mathcal{D}(\Phi)\ \mu\text{-a.e.}\\ \Phi g^{1/2}\in\mathcal{L}^2(\Lambda,\mathcal{A},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0),\mu)\end{cases}$$

(b) If $\Phi,\Psi\in\mathscr{L}^2(\Lambda,\mathcal{A},\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu)$, then (Φ,Ψ) is ν-integrable and

$$\int\Phi\mathrm{d}\nu\Psi^{\mathsf{H}}=\left[\Phi g^{1/2},\Psi g^{1/2}\right]_{L^2(\Lambda,\mathcal{A},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0),\mu)}\,, \tag{5.2}$$

where the latter Gramian comes from (5.1). Hence the mapping $\Phi\mapsto\Phi g^{1/2}$ is Gramian-isometric from $\mathsf{L}^2(\Lambda,\mathcal{A},\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu)$ to $L^2(\Lambda,\mathcal{A},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0),\mu)$.

Proof. Let $f=\frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$. Using that $\|\nu\|_1(\{g=0\})=\int_{\{g=0\}}\|g\|_1\,\mathrm{d}\mu=0$ and $g=f\|g\|_1$ μ-a.e. by uniqueness of the density, we get that

$$\|g\|_1>0\quad\|\nu\|_1\text{-a.e.}\quad\text{and}\quad g=f\|g\|_1\quad\mu\text{-a.e.} \tag{5.3}$$

(and thus also $\|\nu\|_1$-a.e. since $\|\nu\|_1\ll\mu$). From this observation, we get easily that Assertions (i), (ii) and (iii) of Definition 3.4 are respectively equivalent to

(i') We have $\mathrm{Im}(g^{1/2})\subset\mathcal{D}(\Phi)$ and $\mathrm{Im}(g^{1/2})\subset\mathcal{D}(\Psi)$, μ-a.e.

(ii') We have $\Phi g^{1/2}\in\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0)$ and $\Psi g^{1/2}\in\mathcal{S}_2(\mathcal{H}_0,\mathcal{I}_0)$, μ-a.e.

(iii') $(\Phi g^{1/2})(\Psi g^{1/2})^{\mathsf{H}}\in\mathcal{L}^1(\Lambda,\mathcal{A},\mathcal{S}_1(\mathcal{G}_0,\mathcal{I}_0),\mu)$.

We also get easily that

$$\int\Phi\mathrm{d}\nu\Psi^{\mathsf{H}}=\int(\Phi g^{1/2})(\Psi g^{1/2})^{\mathsf{H}}\,\mathrm{d}\mu\,. \tag{5.4}$$

Let us for instance detail the proof of the equivalence between (i') and (i) of Definition 3.4. The left-hand side of (5.3) gives that

$$\|\nu\|_1\left(\left\{\mathrm{Im}(f^{1/2})\not\subset\mathcal{D}(\Phi)\right\}\right)=\|\nu\|_1\left(\left\{\mathrm{Im}(f^{1/2})\not\subset\mathcal{D}(\Phi)\right\}\cap\{g\neq0\}\right)\,, \tag{5.5}$$

and its right-hand side yields

$$\begin{aligned}\mu\left(\left\{\mathrm{Im}(f^{1/2})\not\subset\mathcal{D}(\Phi)\right\}\cap\{g\neq0\}\right)&=\mu\left(\left\{\mathrm{Im}(g^{1/2})\not\subset\mathcal{D}(\Phi)\right\}\cap\{g\neq0\}\right)\\&=\mu\left(\left\{\mathrm{Im}(g^{1/2})\not\subset\mathcal{D}(\Phi)\right\}\right)\,,\end{aligned} \tag{5.6}$$

since $\left\{\mathrm{Im}(g^{1/2})\not\subset\mathcal{D}(\Phi)\right\}\cap\{g=0\}=\emptyset$. To get (i') $\Leftrightarrow$ (i), we note that

$$\|\nu\|_1\left(\left\{\mathrm{Im}(f^{1/2})\not\subset\mathcal{D}(\Phi)\right\}\cap\{g\neq0\}\right)=\int_{\left\{\mathrm{Im}(f^{1/2})\not\subset\mathcal{D}(\Phi)\right\}\cap\{g\neq0\}}\|g\|_1\,\mathrm{d}\mu\,,$$

and thus the right-hand side of (5.5) is zero if and only if the left-hand side of (5.6) is. Now, Assertions (a) and (b) come easily using the definition of $\mathscr{L}^2(\Lambda,\mathcal{A},\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu)$. Note that measurability of $\Phi g^{1/2}$ and $(\Phi g^{1/2})(\Phi g^{1/2})^{\mathsf{H}}$ are ensured by $\mathcal{O}$-measurability of Φ, simple measurability of g and Lemma 5.1. □

We can now derive Theorem 3.1.

Proof of Theorem 3.1. All theses results are easily derived from Proposition 5.3 and the module nature of $L^2(\Lambda,\mathcal{A},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0),\mu)$. The only difficulty lies in showing the completeness of $\mathsf{L}^2(\Lambda,\mathcal{A},\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu)$, which is detailed in the proof of Theorem 11 in [15, Section 3.4]. □

We now provide a useful result about dense subsets of $\mathsf{L}^2(\Lambda,\mathcal{A},\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu)$.

Theorem 5.4. *Let $\mathcal{H}_0,\mathcal{G}_0$ be two separable Hilbert spaces, $(\Lambda,\mathcal{A})$ a measurable space, and ν a trace-class p.o.v.m. on $(\Lambda,\mathcal{A},\mathcal{H}_0)$. Then the space $L^2(\Lambda,\mathcal{A},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\|\nu\|_1)$ is dense in $\mathsf{L}^2(\Lambda,\mathcal{A},\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu)$ and the following assertions hold.*

(i) *The space* $\operatorname{Span}(1_A \mathrm{Q} \ :\ A \in \mathcal{A}, \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0))$ *of simple* $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$*-valued functions is dense in* $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$.

(ii) *For any subset* $E \subset L^2(\Lambda, \mathcal{A}, \|\nu\|_1)$ *which is linearly dense in* $L^2(\Lambda, \mathcal{A}, \|\nu\|_1)$*, the space* $\operatorname{Span}(h \mathrm{Q} \ :\ h \in E, \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0))$ *is dense in* $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$.

Proof. In the first two steps of the proof of Theorem 12 in [15, Section 3.4] (see also [19, Theorem 4.22]), it is shown that, if $\Phi \in \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ and $\epsilon > 0$, there exists $\Psi \in L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1) \subset \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ such that $\|\Phi - \Psi\|_\nu < \epsilon$. This implies that $L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ is dense in $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$. Then Assertion (i) follows using the usual density of simple functions and the fact that, for all $\Phi \in L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$,

$$\begin{aligned} \|\Phi\|_\nu^2 = \operatorname{Tr} \int \Phi\, f\, \Phi^{\mathsf{H}} \,\mathrm{d}\|\nu\|_1 &= \int \operatorname{Tr}\left(\Phi\, f\, \Phi^{\mathsf{H}}\right) \mathrm{d}\|\nu\|_1 \\ &\le \int \|\Phi\|^2_{\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)} \,\mathrm{d}\|\nu\|_1 = \|\Phi\|^2_{L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)}\ , \end{aligned} \tag{5.7}$$

where we used again that $\|f\|_1 = 1$, $\|\nu\|_1$-a.e. Assertion (ii) then follows by approximating, for any $A \in \mathcal{A}$ and $\mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ the function $1_A \mathrm{Q}$ by $g\mathrm{Q}$ with $g \in \operatorname{Span}(E)$ arbitrarily close to 1_A in $L^2(\Lambda, \mathcal{A}, \|\nu\|_1)$. ☐

With this in mind, we can prove Theorem 3.2.

Proof of Theorem 3.2. We set $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ and $\mathcal{G} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$. For all $A, B \in \mathcal{A}$ and $\mathrm{Q}, \mathrm{T} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, we have, by Theorem 3.1,

$$\begin{aligned} [1_A \mathrm{Q}, 1_B \mathrm{T}]_{\nu_W} &= \mathrm{Q} \nu_W(A \cap B) \mathrm{T}^{\mathsf{H}} \\ &= \mathrm{Q} \operatorname{Cov}(W(A), W(B))\, \mathrm{T}^{\mathsf{H}} \\ &= \operatorname{Cov}(\mathrm{Q}W(A), \mathrm{T}W(B)) \\ &= [\mathrm{Q}W(A), \mathrm{T}W(B)]_{\mathcal{G}}\ . \end{aligned}$$

Then Proposition 2.2, applied to $J = \mathcal{A} \times \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ with $v_{(A,\mathrm{Q})} = 1_A \mathrm{Q}$ and $w_{(A,\mathrm{Q})} = \mathrm{Q}W(A)$, gives that there exists a unique Gramian-isometric operator

$$I_W^{\mathcal{G}_0} : \overline{\operatorname{Span}}^{\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)} (1_A \mathrm{T} \mathrm{Q} \ :\ A \in \mathcal{A}, \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \mathrm{T} \in \mathcal{L}_b(\mathcal{G}_0)) \to \mathcal{G} \tag{5.8}$$

such that for all $A \in \mathcal{A}, \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, $I_W^{\mathcal{G}}(1_A \mathrm{Q}) = \mathrm{Q}W(A)$ and, in addition,

$$\operatorname{Im}(I_W^{\mathcal{G}_0}) = \overline{\operatorname{Span}}^{\mathcal{G}} (\mathrm{T}\mathrm{Q}W(A) \ :\ A \in \mathcal{A}, \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \mathrm{T} \in \mathcal{L}_b(\mathcal{G}_0))\ . \tag{5.9}$$

Now, note that

$$\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0) = \{\mathrm{T}\mathrm{Q} \ :\ \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \mathrm{T} \in \mathcal{L}_b(\mathcal{G}_0)\}\ . \tag{5.10}$$

This gives that

$$\operatorname{Span}(1_A \mathrm{T}\mathrm{Q} \ :\ A \in \mathcal{A}, \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \mathrm{T} \in \mathcal{L}_b(\mathcal{G}_0)) = \operatorname{Span}(1_A \mathrm{Q} \ :\ A \in \mathcal{A}, \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0))\ .$$

Therefore, by Theorem 5.4, the domain of $I_W^{\mathcal{G}_0}$ in (5.8) is the whole space $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$. Finally, (5.10) with (5.9) yields

$$\operatorname{Im}(I_W^{\mathcal{G}_0}) = \overline{\operatorname{Span}}^{\mathcal{G}} (\mathrm{Q}W(A) \ :\ A \in \mathcal{A}, \mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)) = \mathcal{H}^{W, \mathcal{G}_0}\ ,$$

which concludes the proof. ☐

5.3 Proofs of Section 4.1

Let us start with the proof of the Gramian-Cramér representation theorem, as a consequence of the Stone theorem. Our proof is mainly a more detailed version of the proof of Theorem 2 in [15, Section 4.2]. However, for completeness, we also prove the uniqueness of $\hat{X}$ and the converse statement (Lemma 4.2) whose proofs are not provided in [15]. The usual Stone theorem (see e.g. [7, Chapter IX]) says that any continuous isomorphism $h \mapsto U_h$ from an l.c.a. group G to the set of unitary operators from a Hilbert space $\mathcal{H}$ onto itself can be represented as an integral of this mapping, that is,

$$U_h = \int \chi(h)\, \xi(\mathrm{d}\chi) ,$$

where ξ is a p.o.v.m. defined on the dual set of characters $\hat{\mathrm{G}}$ endowed with its Borel σ-field and valued in the set of orthogonal projections on $\mathcal{H}$. This classical theorem has a counterpart in the case where $\mathcal{H}$ is an $\mathcal{L}_b(\mathcal{H}_0)$-normal Hilbert module and each U_h is not only unitary but also Gramian-unitary, in which case ξ is valued in the set of orthogonal projections on $\mathcal{H}$ whose ranges are closed submodules. See [15, Section 2.5] for details. It turns out that such p.o.v.m.'s are related to g.o.s. measure by the following lemma.

Lemma 5.5. *Let $\mathcal{H}_0$ be a separable Hilbert space, $\mathcal{H}$ an $\mathcal{L}_b(\mathcal{H}_0)$-normal Hilbert module and $(\Lambda, \mathcal{A})$ a measurable space. Let ξ be a p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H})$ valued in the set of orthogonal projections on $\mathcal{H}$ whose ranges are closed submodules. Then for all $x_0 \in \mathcal{H}$, the mapping $\xi x_0 : A \mapsto \xi(A)x_0$ is a g.o.s. measure on $(\Lambda, \mathcal{A}, \mathcal{H})$ which is regular if ξ is regular.*

Proof. Using the fact that ξ is a p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H})$ and [2, Proposition 1], it is straightforward to see that ξx_0 is an $\mathcal{H}$-valued measure. Moreover, since ξ is valued in the set of orthogonal projections on $\mathcal{H}$ whose ranges are closed submodules, we get that for all disjoint $A, B \in \mathcal{B}(\mathrm{G})$

$$[\xi(A)x_0, \xi(B)x_0]_{\mathcal{H}} = [\xi(B)\xi(A)x_0, x_0]_{\mathcal{H}} = [\xi(B \cap A)x_0, x_0]_{\mathcal{H}} = 0 ,$$

where the first equality is justified in [15, P. 23] and the second one by [2, Theorem 3]. This proves that ξx_0 is a g.o.s. measure on $(\Lambda, \mathcal{A}, \mathcal{H})$. In the following, we denote by ν its intensity operator measure. Then, for all $A \in \mathcal{A}$, we have

$$\|\nu(A)\|_1 = \mathrm{Tr}[\xi(A)x_0, \xi(A)x_0]_{\mathcal{H}} = \langle \xi(A)x_0, x_0 \rangle_{\mathcal{H}} ,$$

where the last equality comes from the fact that $\xi(A)$ is an orthogonal projection on $\mathcal{H}$. Now, if ξ is regular, then the measure $A \mapsto \langle \xi(A)x_0, x_0 \rangle$ is regular and so is $\|\nu\|_1$ by the previous display. This implies that ξx_0 is regular and the proof is concluded. □

The proof of uniqueness in Theorem 4.1 requires the following result which is a kind of Fubini theorem for interchanging a Bochner integral with a g.o.s. integral.

Proposition 5.6. *Let $(\Lambda, \mathcal{A})$ be a measurable space and $\mathcal{H}_0$, $\mathcal{G}_0$ two separable Hilbert spaces. Let W be an $\mathcal{H}_0$-valued random g.o.s. measure on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure ν_W. Let μ be a non-negative measure on a measurable space $(\Lambda', \mathcal{A}')$. Suppose that Φ is measurable from $\Lambda \times \Lambda'$ to $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and satisfies*

$$\int \left(\int \|\Phi(\lambda, \lambda')\|_{\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)} \, \mu(\mathrm{d}\lambda') \right)^2 \|\nu_W\|_1(\mathrm{d}\lambda) < \infty , \tag{5.11}$$

$$\int \left(\int \|\Phi(\lambda, \lambda')\|^2_{\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)} \, \|\nu_W\|_1(\mathrm{d}\lambda) \right)^{1/2} \mu(\mathrm{d}\lambda') < \infty . \tag{5.12}$$

Then we have

$$\int \left(\int \Phi(\lambda, \lambda') \, \mu(\mathrm{d}\lambda') \right) W(\mathrm{d}\lambda) = \int \left(\int \Phi(\lambda, \lambda') \, W(\mathrm{d}\lambda) \right) \mu(\mathrm{d}\lambda') , \tag{5.13}$$

where integrals with respect to W are as in Definition 3.5, in the left-hand side the innermost integral is understood as a Bochner integral on $L^2(\Lambda', \mathcal{A}', \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \mu)$ and in the right-hand side, the outermost integral is understood as a Bochner integral on $L^2(\Lambda', \mathcal{A}', \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P}), \mu)$.

Proof. Conditions (5.11) and (5.12) ensure that $\Phi(\lambda, \cdot) \in L^1(\Lambda', \mathcal{A}', \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \mu)$ for $\|\nu_W\|_1$-a.e. $\lambda \in \Lambda$ and that $\Phi(\cdot, \lambda') \in L^2(\Lambda, \mathcal{A}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ for μ-a.e. $\lambda' \in \Lambda'$, showing that the innermost integrals in both sides of (5.13) are well defined for adequate sets of λ and λ', respectively. Let E_1 and E_2 denote the sets of functions Φ measurable from $\Lambda \times \Lambda'$ to $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and satisfying (5.11) and (5.12), respectively. We denote by $\|\Phi\|_{E_1}$ the square root of the left-hand side of (5.11) and by $\|\Phi\|_{E_2}$ the left-hand side of (5.12), which make E_1 and

E_2 Banach spaces. Then, for all $\Phi \in E := E_1 \cap E_2$, concerning the left-hand side of (5.13), we have

$$\left\| \int \Phi(\cdot, \lambda') \, \mu(\mathrm{d}\lambda') \right\|_{\nu_W}^2 \leq \int \left\| \int \Phi(\cdot, \lambda') \, \mu(\mathrm{d}\lambda') \right\|_{\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)}^2 \mathrm{d}\|\nu_W\|_1 \leq \|\Phi\|_{E_1}^2 ,$$

as for the right-hand side, we have, setting $\mathcal{H} := \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$,

$$\int \left\| \int \Phi(\lambda, \cdot) \, W(\mathrm{d}\lambda) \right\|_{\mathcal{H}} \mathrm{d}\mu = \int \|\Phi(\cdot, \lambda')\|_{\nu_W} \, \mu(\mathrm{d}\lambda') \leq \|\Phi\|_{E_2} ,$$

These two inequalities show that both sides of (5.13) seen as functions of Φ are linear continuous from E endowed with the norm $\|\cdot\|_E = \|\cdot\|_{E_1} + \|\cdot\|_{E_2}$ to $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$. Since they coincide for $\Phi(\lambda, \lambda') = 1_A(\lambda) 1_B(\lambda') \mathrm{Q}$ with $A \in \mathcal{A}$, $B \in \mathcal{A}'$ and $\mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, this concludes the proof. □

An interesting straightforward application of Proposition 5.6 is the following extension of Example 4.1.

Example 5.1 (Convolutional filtering)**.** *Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces. Let $X = (X_t)_{t \in \mathrm{G}}$ be an $\mathcal{H}_0$-valued weakly stationary stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let η be the Haar measure on G (see [25, Chapter 1]) and $\Phi \in L^1(\mathrm{G}, \mathcal{B}(\mathrm{G}), \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \eta)$. Define the process $Y = (Y_t)_{t \in \mathrm{G}}$ by the* time domain convolutional filtering

$$Y_t = \int \Phi(s) \, X_{t-s} \, \eta(\mathrm{d}s) , \quad t \in \mathrm{G} ,$$

where the integral is a Bochner integral on $L^1(\mathrm{G}, \mathcal{B}(\mathrm{G}), \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P}), \eta)$. Then, defining $\hat{\Phi} : \hat{\mathrm{G}} \to \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ by the following Bochner integral on $L^1(\mathrm{G}, \mathcal{B}(\mathrm{G}), \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \eta)$,

$$\hat{\Phi}(\chi) = \int \Phi(s) \, \overline{\chi(s)} \, \eta(\mathrm{d}s) ,$$

we can show that $\hat{\Phi} \in \mathsf{L}^2(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_X)$ and $Y = F_{\hat{\Phi}}(X)$.

We can now provide a detailed proof of Theorem 4.1.

Proof of Theorem 4.1. Suppose that X is weakly stationary as in Definition 4.1. Then the collection of lag operators $(U_h^X)_{h \in \mathrm{G}}$ satisfies the assumptions of the generalized Stone's theorem stated as Proposition 4 in [15, Section 2.5]. This gives that there exists a regular p.o.v.m. ξ^X on $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{H}^X)$ valued in the set of orthogonal projections whose ranges are closed submodules of $\mathcal{H}^X$ such that, for all $h \in \mathrm{G}$,

$$U_h^X = \int \chi(h) \, \xi^X(\mathrm{d}\chi) , \tag{5.14}$$

where the integral is as in Theorem 9 in [2]. Then, by Lemma 5.5, the mapping

$$\hat{X} : \begin{array}{ccc} \mathcal{B}(\hat{\mathrm{G}}) & \to & \mathcal{H}^X \\ A & \mapsto & \xi^X(A) X_0 \end{array} \tag{5.15}$$

is a regular g.o.s. measure on $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{H}^X)$ and we denote by ν_X its intensity operator measure. Since $\mathcal{H}^X$ is a submodule of $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$, $\hat{X}$ is also a regular $\mathcal{H}_0$-valued random g.o.s. measure on $(\Omega, \mathcal{F}, \mathbb{P})$, see Definition 3.2. Relation (4.2) then follows by applying (5.14) and the fact that, for all $t \in \mathrm{G}$, $U_t^h X_0 = X_t$ and, for all $\phi : \Lambda \to \mathbb{C}$ measurable and bounded,

$$\int \phi \, \mathrm{d}\hat{X} = \left(\int \phi \, \mathrm{d}\xi^X \right) X_0 , \tag{5.16}$$

where the integral in the left-hand side is defined as in Definition 3.5 (see also Remark 3.2) and the integral in the right-hand side as in Theorem 9 in [2], for the p.o.v.m. ξ^X. Relation (5.16) obviously holds if $\phi = 1_A$ with $A \in \mathcal{A}$ and also for ϕ simple by linearity. Now, for a general measurable and bounded $\phi : \Lambda \to \mathbb{C}$, we can find a sequence $(\phi_n)_{n \in \mathbb{N}}$ of simple functions such that $|\phi_n| \leq |\phi|$ for all $n \in \mathbb{N}$ and $\phi_n(\lambda) \to \phi(\lambda)$ as $n \to \infty$ for all $\lambda \in \Lambda$. Then, by dominated convergence, ϕ_n converges to ϕ in $L^2(\Lambda, \mathcal{A}, \|\nu\|_1)$ and therefore $\phi_n \mathrm{Id}$ converges to $\phi\mathrm{Id}$ in $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0), \nu)$. Thus $\int \phi_n \, \mathrm{d}\hat{X} \to \int \phi \, \mathrm{d}\hat{X}$ in $\mathcal{H}^X$ by the isometric property of the integral of Definition 3.5. To get (5.16), it now suffices to show that, for all $Y \in \mathcal{H}^X$, $\left\langle \left(\int \phi_n \, \mathrm{d}\xi\right) X_0, Y \right\rangle_{\mathcal{H}^X} \to \left\langle \left(\int \phi \, \mathrm{d}\xi\right) X_0, Y \right\rangle_{\mathcal{H}^X}$. This follows from the polarization formula, Theorem 9 in [2] and dominated convergence.

To show uniqueness, suppose there exists another regular $\mathcal{H}_0$-valued random g.o.s. measure W on $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \Omega, \mathcal{F}, \mathbb{P})$ satisfying the same identity as (4.2) with $\hat{X}$ replaced by W. Then, we get

$$\int \chi(t) \, \hat{X}(\mathrm{d}\chi) = \int \chi(t) \, W(\mathrm{d}\chi) \quad \text{for all} \quad t \in \mathrm{G} \,. \tag{5.17}$$

Let η denote the Haar measure on G and denote by $\mathcal{C}_c(\mathrm{G})$ the space of compactly supported functions from G to $\mathbb{C}$. Then, by [25, Theorem 1.2.4] and [25, Section E.8], the space

$$E = \left\{ \hat{\phi} : \chi \mapsto \int \phi(t) \overline{\chi(t)} \, \eta(\mathrm{d}t) \; : \; \phi \in L^1(\mathrm{G}, \mathcal{B}(\mathrm{G}), \eta) \right\}$$

is dense in $L^2(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \|\nu_W\|_1 + \|\nu_X\|_1)$. We can thus find, for any $A \in \mathcal{B}(\hat{\mathrm{G}})$, $(\phi_n)_{n \in \mathbb{N}} \in \mathcal{C}_c(\mathrm{G})^{\mathbb{N}}$ such that, defining $\hat{\phi}_n$ as above, $\hat{\phi}_n \to \mathbb{1}_A$ both in $L^2(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \|\nu_W\|_1)$ and in $L^2(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \|\nu_X\|_1)$. Then by Proposition 5.6, we have, for all $n \in \mathbb{N}$,

$$\begin{aligned}\int \hat{\phi}_n(\chi) \, W(\mathrm{d}\chi) &= \int \left(\int \chi(-t) \, W(\mathrm{d}\chi) \right) \phi_n(t) \, \eta(\mathrm{d}t) \\ &= \int \left(\int \chi(-t) \, \hat{X}(\mathrm{d}\chi) \right) \phi_n(t) \, \eta(\mathrm{d}t) = \int \hat{\phi}_n(\chi) \, \hat{X}(\mathrm{d}\chi) \,,\end{aligned}$$

where we have used (5.17) in the second equality. Letting $n \to \infty$, we get $W(A) = \hat{X}(A)$, thus proving the uniqueness of $\hat{X}$. □

We now prove Lemma 4.2.

Proof of Lemma 4.2. By Definition 3.5, $X = (X_t)_{t \in \mathrm{G}}$ is a centered $\mathcal{H}_0$-valued process satisfying (i) and (ii) in Definition 4.1. Using the Gramian-isometric property of integration with respect to W, we get for all $t, h \in \mathrm{G}$, $\mathrm{Cov}\,(X_{t+h}, X_t) = \int \chi(t+h)\overline{\chi(t)} \nu_X(\mathrm{d}\chi) = \int \chi(h) \nu_X(\mathrm{d}\chi)$ which gives (iii) in Definition 4.1 with autocovariance operator function Γ given by (4.3). Finally, for all $\mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0)$, for all $h \in \mathrm{G}$, denoting by f the density of ν with respect to $\|\nu\|_1$, we have

$$\mathrm{Q}\Gamma(h) = \mathrm{Q} \int \chi(h) \, f(\chi) \, \|\nu\|_1(\mathrm{d}\chi) = \int \chi(h) \, \mathrm{Q} \, f(\chi) \, \|\nu\|_1(\mathrm{d}\chi) \,,$$

Since the integrand in the last integral has trace-class norm upper bounded by $\|\mathrm{Q}\|_{\mathcal{L}_b(\mathcal{H}_0)}$ and $\|\nu\|_1$ is finite we get that $h \mapsto \mathrm{Q}\Gamma(h)$ is continuous from $\hat{\mathrm{G}}$ to $\mathcal{S}_1(\mathcal{H}_0)$ by dominated convergence. The continuity of $h \mapsto \mathrm{Tr}(\mathrm{Q}\Gamma(h))$ follows, thus showing the last point of Definition 4.1. □

We can now prove the Kolmogorov isomorphism theorem.

Proof of Theorem 4.3. By Theorem 3.2 and (4.4), $I_{\hat{X}}^{\mathcal{G}_0}$ is a Gramian-unitary operator from $\widehat{\mathcal{H}}^{X,\mathcal{G}_0}$ to $\mathcal{H}^{\hat{X},\mathcal{G}_0}$. Thus to conclude, we only need to show that $\mathcal{H}^{X,\mathcal{G}_0} = \mathcal{H}^{\hat{X},\mathcal{G}_0}$. By (4.2), we have for all $\mathrm{Q} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and $t \in \mathrm{G}$, $\mathrm{Q}X_t = I_{\hat{X}}^{\mathcal{G}_0}(\mathrm{Q}\mathrm{e}_t) \in \mathcal{H}^{\hat{X},\mathcal{G}_0}$, where $\mathrm{e}_t : \chi \mapsto \chi(t)$. Thus, by (4.1), we get that $\mathcal{H}^{X,\mathcal{G}_0} \subset \mathcal{H}^{\hat{X},\mathcal{G}_0}$. The definition of $\hat{X}$ in (5.15) gives the converse inclusion, which achieves the proof. □

The proof of Corollary 4.5 relies on the following result providing a way to build a g.o.s. measure W from its intensity measure.

Theorem 5.7. *Let $\mathcal{H}_0$ be a separable Hilbert space and $(\Lambda, \mathcal{A})$ be a measurable space. Let ν be a trace-class p.o.v.m. on $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. Then there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an $\mathcal{H}_0$-valued random g.o.s. W on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure ν such that the process $(\langle W(A), x\rangle)_{A \in \mathcal{A}, x \in \mathcal{H}_0}$ is a (complex) Gaussian process.*

Proof. Define $\gamma : (\mathcal{H}_0 \times \mathcal{A})^2 \to \mathbb{C}$ by of

$$\gamma((x, A); (y, B)) = x^{\mathsf{H}} \nu(A \cap B) y = \left[x^{\mathsf{H}} 1_A, y^{\mathsf{H}} 1_B\right]_{\nu} ,$$

where we used the Gramian (3.4) with $\mathcal{G}_0 = \mathbb{C}$. Then it is easy to see γ is hermitian non-negative definite in the sense that for all $n \geq 1$, $x_1, \ldots, x_n \in \mathcal{H}_0$, $A_1, \ldots, A_n \in \mathcal{A}$ and $a_1, \ldots, a_n \in \mathbb{C}$,

$$\sum_{i,j=1}^{n} a_i \overline{a_j} \gamma((x_i, A_i); (a_j, A_j)) \geq 0 .$$

Let $(Z_{x,A})_{(x,A) \in \mathcal{H}_0 \times \mathcal{A}}$ be the centered circularly symmetric Gaussian process complex with covariance γ. Let $(\phi_n)_{0 \leq n < N}$ be a Hilbert basis of $\mathcal{H}_0$, with $N = \dim \mathcal{H}_0 \in \{1, 2, \ldots, \infty\}$. It is straightforward to show that for all $A \in \mathcal{A}$,

$$W(A) := \sum_{0 \leq n < N} Z_{\phi_n, A}\, \phi_n$$

is well defined in $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ and that the so defined W is a random g.o.s. with intensity operator measure ν. □

We can now prove Corollary 4.5.

Proof of Corollary 4.5. The implication (i)⇒(ii) follows from Theorem 4.1. Now suppose that (ii) holds. Let W be the Gaussian g.o.s. measure with intensity operator measure ν obtained in Theorem 5.7. Then Assertion (i) follows from Lemma 4.2. □

The proof of Theorem 4.6 is mainly contained in [3].

Proof of Theorem 4.6. The equivalence between (i) and (ii) is straightforward: to show that (i)⇒(ii), take an arbitrary $x \in \mathcal{H}_0$ with unit norm and set $\mathrm{Q}_i = x\, x_i^{\mathsf{H}}$ for $i = 1, \ldots, n$. To show that (ii)⇒(i), take, for any $x \in \mathcal{H}_0$, $x_i = \mathrm{Q}_i^{\mathsf{H}} x$ for $i = 1, \ldots, n$. The equivalence between (ii), (iii) and (iv) is given by [3, Theorem 3]. The lastly stated fact that ν is uniquely determined by (4.3) is a consequence of the uniqueness stated in the univariate Bochner theorem (recalled in Theorem 4.4) applied to $\nu_x : A \mapsto x^{\mathsf{H}} \nu(A)\, x$ for all $x \in \mathcal{H}_0$.

Note that the proof of the implication (iv) ⇒ (ii) provided in [3, Theorem 3] uses dilation theory. We hereafter propose an alternative and more elementary proof of this implication. Suppose that (iv) holds. The continuity of Γ in w.o.t. follows immediately by dominated convergence and we now prove that it is of positive type as in Definition 4.5. Take some arbitrary $n \in \mathbb{N}^*$, and $x_1, \cdots, x_n \in \mathcal{H}_0$. Let us define the $\mathbb{C}^{n \times n}$-valued measure μ on on $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}))$ by

$$\mu(A) = \begin{bmatrix} \langle \nu(A) x_1, x_1 \rangle_{\mathcal{H}_0} & \cdots & \langle \nu(A) x_n, x_1 \rangle_{\mathcal{H}_0} \\ \vdots & \ddots & \vdots \\ \langle \nu(A) x_1, x_n \rangle_{\mathcal{H}_0} & \cdots & \langle \nu(A) x_n, x_n \rangle_{\mathcal{H}_0} \end{bmatrix} .$$

Then, by the Cauchy-Schwartz inequality, for all $i, j \in [\![1, n]\!]$, the $\mathbb{C}$-valued measure $\mu_{i,j} : A \mapsto [\mu(A)]_{i,j}$ admits a density $f_{i,j}$ with respect to the non-negative finite measure $\|\mu\|_1 : A \mapsto \|\mu(A)\|_1 = \mathrm{Tr}(\mu(A))$ and the matrix-valued function $f : \chi \mapsto (f_{i,j}(\chi))_{1 \leq i,j \leq n}$ is $\|\mu\|_1$-a.e. hermitian, non-negative semi-definite since, for all $a \in \mathbb{C}^n$ and $A \in \mathcal{B}(\hat{\mathrm{G}})$,

$$\int_A a^{\mathsf{H}} f(\chi) a \, \|\mu\|_1(\mathrm{d}\chi) = a^{\mathsf{H}} \mu(A) a = \left(\sum_{i=1}^{n} a_i x_i\right)^{\mathsf{H}} \nu(A) \left(\sum_{i=1}^{n} a_i x_i\right) \geq 0 .$$

Then, for all $t_1, \cdots, t_n \in \mathrm{G}$, we have

$$\begin{aligned} \sum_{i,j=1}^{n} \langle \Gamma(t_i - t_j) x_i, x_j \rangle_{\mathcal{H}_0} &= \sum_{i,j=1}^{n} \int \chi(t_i) \overline{\chi(t_j)}\, \mu_{i,j}(\mathrm{d}\chi) = \sum_{i,j=1}^{n} \int \chi(t_i) \overline{\chi(t_j)} f_{i,j}(\chi)\, \|\mu\|_1(\mathrm{d}\chi) \\ &= \int \underbrace{\left(\sum_{i,j=1}^{n} \chi(t_i) \overline{\chi(t_j)} f_{i,j}(\chi)\right)}_{\geq 0\ \|\mu\|_1\text{-a.e.}} \|\mu\|_1(\mathrm{d}\chi) \\ &\geq 0 . \end{aligned}$$

The first line follows from (iv), the definition of $\mu_{i,j}$ above and the definition of the integral as given by Theorem 9 in [2]. The second line follows from the definition of $f_{i,j}$ and the third line from the above property of the matrix-valued function f. Hence we have shown (ii) and the proof of the implication is concluded. □

We conclude this section with the proof of Corollary 4.7

Proof of Corollary 4.7. By definition of the autocovariance operator function of a weakly stationary process, it is straightforward to see that Assertion (i) implies Assertion (ii). Now, suppose that Assertion (ii) holds. By Corollary 4.5, we only need to prove Assertion (ii) of Corollary 4.5, which is what we almost get in Assertion (iv) of Theorem 4.6, except that we have to prove that, additionally, ν is trace-class. Applying (4.3) with $h = 0$, we get that $\nu(\hat{\mathrm{G}}) = \Gamma(0)$, which is assumed to be in $\mathcal{S}_1(\mathcal{H}_0)$ in the present Assertion (ii). Thus by Lemma 2.1, ν is indeed trace-class and the proof is concluded. □

5.4 Filtering random g.o.s. measures and proofs of Section 4.2

Having a clear description of the modular spectral domain at hand, the results of Section 4.2, mainly Proposition 4.8, can be seen as a particular instance of the composition and inversion of operator-valued functions filtering a general random g.o.s. measure, which is the framework of this section. We first state a straightforward result, whose proof is omitted.

Proposition 5.8. *Let $(\Lambda, \mathcal{A})$ be a measurable space, $\mathcal{H}_0$, $\mathcal{G}_0$ two separable Hilbert spaces. Let W be an $\mathcal{H}_0$-valued random g.o.s. measure on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure ν_W. Let $\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$. Then the mapping*

$$V : A \mapsto \int_A \Phi \,\mathrm{d}W = I_W^{\mathcal{G}_0}(1_A \Phi) \tag{5.18}$$

is a $\mathcal{G}_0$-valued random g.o.s. measure on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure

$$\Phi \nu_W \Phi^{\mathsf{H}} : A \mapsto \int_A \Phi \mathrm{d}\nu_W \Phi^{\mathsf{H}} \;,$$

which is a well defined trace-class p.o.v.m.

The g.o.s. V defined by (5.18) is said to admit the density Φ with respect to W, and we write $\mathrm{d}V = \Phi \mathrm{d}W$ (or, equivalently, $V(\mathrm{d}\lambda) = \Phi(\lambda) W(\mathrm{d}\lambda)$). In the following definition, based on Proposition 5.8, we use a signal processing terminology where Λ is seen as a set of frequencies and Φ is seen as a transfer operator function acting on the (random) input frequency distribution W.

Definition 5.1 (Filter $\hat{F}_\Phi(W)$ acting on a random g.o.s. measure in $\hat{\mathcal{S}}_\Phi$)**.** *Let $(\Lambda, \mathcal{A})$ be a measurable space, $\mathcal{H}_0$, $\mathcal{G}_0$ two separable Hilbert spaces. For a given transfer operator function $\Phi \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$, we denote by $\hat{\mathcal{S}}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$ the set of $\mathcal{H}_0$-valued random g.o.s. measures on $(\Lambda, \mathcal{A}, \Omega, \mathcal{F}, \mathbb{P})$ whose intensity operator measures ν_W satisfy $\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$. Then, for any $W \in \hat{\mathcal{S}}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$, we say that the random $\mathcal{G}_0$-valued g.o.s. measure V defined by (5.18) is the output of the filter with transfer operator function Φ applied to the input g.o.s. measure W, and we denote $V = \hat{F}_\Phi(W)$.*

Now, consider the filtering, using Definition 5.1, $V = \hat{F}_\Phi(W)$ for a random g.o.s. measure W and a transfer function $\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$. The goal of this section is, given another separable Hilbert space $\mathcal{I}_0$, to characterize the transfer functions Ψ valued in $\mathcal{O}(\mathcal{G}_0, \mathcal{I}_0)$ which can be used to filter the g.o.s. measure V. Taking W to be the Cramér representation $\hat{X}$ of a weakly stationary process X, we will get the already stated Proposition 4.8 on the composition of linear filters as a byproduct.

According to Proposition 5.8, Ψ must be square-integrable with respect to $\nu_V = \Phi \nu_W \Phi^{\mathsf{H}}$ and this turns out to be equivalent to checking that $\Psi\Phi$ is square integrable with respect to ν_W as stated in the following theorem. We recall that $\Psi\Phi$ is the pointwise composition, that is, $\Psi\Phi : \lambda \mapsto \Psi(\lambda) \circ \Phi(\lambda)$ and is defined whenever the image of $\Phi(\lambda)$ is included in the domain of $\Psi(\lambda)$. We first need the following lemma, which will be used in the proof of Theorem 5.10.

Lemma 5.9. *Let $\mathcal{H}_0$, $\mathcal{G}_0$, $\mathcal{I}_0$ be separable Hilbert spaces and $\mathrm{Q} \in \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0)$, $\mathrm{T} \in \mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$. The following assertions hold.*

(i) $\mathrm{Im}(|\mathrm{T}^{\mathsf{H}}|) = \mathrm{Im}(\mathrm{T})$.

(ii) *If* $\mathrm{Im}(\mathrm{T}) \subset \mathcal{D}(\mathrm{Q})$, *then* $(\mathrm{QT})(\mathrm{QT})^{\mathsf{H}} = (\mathrm{Q}\,|\mathrm{T}^{\mathsf{H}}|)(\mathrm{Q}\,|\mathrm{T}^{\mathsf{H}}|)^{\mathsf{H}}$.

(iii) *If* $\mathrm{Im}(\mathrm{T}) \subset \mathcal{D}(\mathrm{Q})$, *then* $\mathrm{QT} \in \mathcal{S}_2(\mathcal{H}_0, \mathcal{I}_0)$ *if and only if* $\mathrm{Q}\,|\mathrm{T}^{\mathsf{H}}| \in \mathcal{S}_2(\mathcal{G}_0, \mathcal{I}_0)$. *In this case* $\|\mathrm{QT}\|_2 = \|\mathrm{Q}\,|\mathrm{T}^{\mathsf{H}}|\|_2$.

Proof. For convenience, we only consider the case where the spaces have infinite dimensions. The singular values decomposition of T yields for two orthonormal sequences $(\psi_n)_{n\in\mathbb{N}} \in \mathcal{G}_0^{\mathbb{N}}$ and $(\phi_n)_{n\in\mathbb{N}} \in \mathcal{H}_0^{\mathbb{N}}$,

$$\mathrm{T} = \sum_{n\in\mathbb{N}} \sigma_n \psi_n \otimes \phi_n \quad \text{and} \quad \left|\mathrm{T}^{\mathsf{H}}\right| = \sum_{n\in\mathbb{N}} \sigma_n \psi_n \otimes \psi_n \ .$$

Proof of (i). We have $\mathrm{Im}(\mathrm{T}) = \left\{\sum_{n\in\mathbb{N}} \sigma_n x_n \psi_n \ : \ (x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})\right\} = \mathrm{Im}(|\mathrm{T}^{\mathsf{H}}|)$.

Proof of (ii). By the first point both compositions QT and $\mathrm{Q}\,|\mathrm{T}^{\mathsf{H}}|$ make sense. Consider the polar decomposition of T^{H} : $\mathrm{T}^{\mathsf{H}} = U\,|\mathrm{T}^{\mathsf{H}}|$, with $U = \sum_{n\in\mathbb{N}} \phi_n \otimes \psi_n$. Then $\mathrm{T} = |\mathrm{T}^{\mathsf{H}}|\, U^{\mathsf{H}}$ and

$$(\mathrm{QT})(\mathrm{QT})^{\mathsf{H}} = \left(\mathrm{Q}\left|\mathrm{T}^{\mathsf{H}}\right|\right) U^{\mathsf{H}} U \left(\mathrm{Q}\left|\mathrm{T}^{\mathsf{H}}\right|\right)^{\mathsf{H}} = \left(\mathrm{Q}\left|\mathrm{T}^{\mathsf{H}}\right|\right)\left(\mathrm{Q}\left|\mathrm{T}^{\mathsf{H}}\right|\right)^{\mathsf{H}} ,$$

where we used that $|\mathrm{T}^{\mathsf{H}}|\, U^{\mathsf{H}} U = |\mathrm{T}^{\mathsf{H}}|$.

Proof of (iii). We have that $\mathrm{QT} \in \mathcal{S}_2(\mathcal{H}_0, \mathcal{I}_0)$ if and only if $(\mathrm{QT})(\mathrm{QT})^{\mathsf{H}} \in \mathcal{S}_1(\mathcal{I}_0)$, which is equivalent to $\mathrm{Q}\,|\mathrm{T}^{\mathsf{H}}| \in \mathcal{S}_2(\mathcal{G}_0, \mathcal{I}_0)$ by the previous point. □

We can now derive the main result of this section.

Theorem 5.10. *Let* $(\Lambda, \mathcal{A})$ *be a measurable space,* $\mathcal{H}_0$, $\mathcal{G}_0$, $\mathcal{I}_0$ *separable Hilbert spaces and* ν *a trace-class p.o.v.m. on* $(\Lambda, \mathcal{A}, \mathcal{H}_0)$. *Let* $\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ *and* $\Psi \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{G}_0, \mathcal{I}_0)$. *Define* $\Phi\nu\Phi^{\mathsf{H}} : A \mapsto \int_A \Phi \mathrm{d}\nu \Phi^{\mathsf{H}} = [1_A \Phi, 1_A \Phi]_\nu$, *which is a trace-class p.o.v.m. on* $(\Lambda, \mathcal{A}, \mathcal{G}_0)$. *Then*

$$\Psi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}}) \Leftrightarrow \Psi\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \nu) \ . \tag{5.19}$$

Moreover, the following assertions hold.

(a) *For all* $\Psi, \Theta \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}})$, *we have* $(\Psi\Phi)\nu(\Theta\Phi)^{\mathsf{H}} = \Psi(\Phi\nu\Phi^{\mathsf{H}})\Theta^{\mathsf{H}}$.

(b) *The mapping* $\Psi \mapsto \Psi\Phi$ *is a well defined Gramian-isometric operator from* $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}})$ *to* $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \nu)$.

(c) *If moreover* Φ *is injective* $\|\nu\|_1$*-a.e., then* $\Phi^{-1} \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{H}_0), \Phi\nu\Phi^{\mathsf{H}})$, *where we define* $\Phi^{-1}(\lambda) := \left(\Phi(\lambda)_{|\mathcal{D}(\Phi(\lambda))\to\mathrm{Im}(\Phi(\lambda))}\right)^{-1}$ *with domain* $\mathrm{Im}(\Phi(\lambda))$ *for all* $\lambda \in \{\Phi$ *is injective*$\}$ *and* $\Phi^{-1}(\lambda) = 0$ *otherwise.*

Proof. Let μ be a dominating measure for $\|\nu\|_1$ and $g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$, then, by definition of $\Phi\nu\Phi^{\mathsf{H}}$, μ also dominates $\|\Phi\nu\Phi^{\mathsf{H}}\|_1$ and $\frac{\mathrm{d}\Phi\nu\Phi^{\mathsf{H}}}{\mathrm{d}\mu} = (\Phi g^{1/2})(\Phi g^{1/2})^{\mathsf{H}}$. Hence, $\left(\frac{\mathrm{d}\Phi\nu\Phi^{\mathsf{H}}}{\mathrm{d}\mu}\right)^{1/2} = \left|(\Phi g^{1/2})^{\mathsf{H}}\right|$ and we get, by Proposition 5.3,

$$\begin{aligned}
\Psi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}}) &\Leftrightarrow \begin{cases} \mathrm{Im}\left|(\Phi g^{1/2})^{\mathsf{H}}\right| \subset \mathcal{D}(\Psi) \ \ \mu\text{-a.e.} \\ \Psi\left|(\Phi g^{1/2})^{\mathsf{H}}\right| \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{G}_0, \mathcal{I}_0), \mu) \end{cases} \\
&\Leftrightarrow \begin{cases} \mathrm{Im} g^{1/2} \subset \mathcal{D}(\Psi\Phi) \ \ \mu\text{-a.e.} \\ \Psi\Phi g^{1/2} \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{I}_0), \mu) \end{cases} \\
&\Leftrightarrow \Psi\Phi \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \nu) ,
\end{aligned}$$

where the second equivalence comes from Lemma 5.9 and the fact that for all $\lambda \in \Lambda$, $\mathcal{D}(\Psi(\lambda)\Phi(\lambda))$ is the preimage of $\mathcal{D}(\Psi(\lambda))$ by $\Phi(\lambda)$ which gives that $\mathrm{Im}(g^{1/2}(\lambda)) \subset \mathcal{D}(\Psi(\lambda)\Phi(\lambda))$ if and only if $\mathrm{Im}(\Phi(\lambda)g^{1/2}(\lambda)) \subset \mathcal{D}(\Psi(\lambda))$.

We now prove Assertion (a). For $\Psi, \Theta \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}})$ and $A \in \mathcal{A}$, we have

$$\begin{aligned}
(\Psi\Phi)\nu(\Theta\Phi)^{\mathsf{H}}(A) = \int_A \left(\Psi\Phi g^{1/2}\right)\left(\Theta\Phi g^{1/2}\right)^{\mathsf{H}} \mathrm{d}\mu &= \int_A \left(\Psi\left|(\Phi g^{1/2})^{\mathsf{H}}\right|\right)\left(\Theta\left|(\Phi g^{1/2})^{\mathsf{H}}\right|\right)^{\mathsf{H}} \mathrm{d}\mu \\
&= \Psi(\Phi\nu\Phi^{\mathsf{H}})\Theta^{\mathsf{H}}(A) ,
\end{aligned}$$

where the second equality holds by Lemma 5.9. Assertion (a) follows as well as Assertion (b) by taking $A = \Lambda$. Finally, to show Assertion (c), suppose that Φ is injective $\|\nu\|_1$-a.e. then $\Phi^{-1}\Phi : \lambda \mapsto \mathrm{Id}_{\mathcal{H}_0} \mathbb{1}_{\{\Phi(\lambda) \text{ is injective}\}}$ is in $\mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0), \nu)$ which gives that $\Phi^{-1} \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{H}_0), \Phi\nu\Phi^{\mathsf{H}})$ by Assertion (a). □

We deduce the following corollary on the composition and inversion for random g.o.s. measures.

Corollary 5.11 (Composition and inversion of filters on random g.o.s. measures). *Let $(\Lambda, \mathcal{A})$ be a measurable space, $\mathcal{H}_0$, $\mathcal{G}_0$ two separable Hilbert spaces, and $\Phi \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{H}_0, \mathcal{G}_0)$. Let $W \in \hat{\mathcal{S}}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$ with intensity operator measure ν_W. Then three following assertions hold.*

(i) For any separable Hilbert space $\mathcal{I}_0$, we have $\mathcal{H}^{\hat{F}_\Phi(W), \mathcal{I}_0} \underset{\approx}{\subseteq} \mathcal{H}^{W, \mathcal{I}_0}$.

(ii) For any separable Hilbert space $\mathcal{I}_0$ and all $\Psi \in \mathbb{F}_{\mathcal{O}}(\Lambda, \mathcal{A}, \mathcal{G}_0, \mathcal{I}_0)$, we have $W \in \hat{\mathcal{S}}_{\Psi\Phi}(\Omega, \mathcal{F}, \mathbb{P})$ if and only if $\hat{F}_\Phi(W) \in \hat{\mathcal{S}}_\Psi(\Omega, \mathcal{F}, \mathbb{P})$, and in this case, we have

$$\hat{F}_\Psi \circ \hat{F}_\Phi(W) = \hat{F}_{\Psi\Phi}(W). \tag{5.20}$$

(iii) Suppose that Φ is injective $\|\nu_W\|_1$-a.e. Then $W = F_{\Phi^{-1}} \circ F_\Phi(W)$, where Φ^{-1} is defined as in Assertion (c) of Theorem 5.10. Moreover, Assertion (i) above holds with $\underset{\approx}{\subseteq}$ replaced by $\cong$.

Proof. **Proof of Assertion (i).** This follows from Assertion (b) of Theorem 5.10 and Theorem 3.2.

Proof of Assertion (ii). If $W \in \hat{\mathcal{S}}_\Phi(\Omega, \mathcal{F}, \mathbb{P})$, then the equivalence between $W \in \hat{\mathcal{S}}_{\Psi\Phi}(\Omega, \mathcal{F}, \mathbb{P})$ and $\hat{F}_\Phi(W) \in \hat{\mathcal{S}}_\Psi(\Omega, \mathcal{F}, \mathbb{P})$ is just another formulation of the equivalence (5.19) with $\nu = \nu_W$. Suppose that it holds and set $V := \hat{F}_\Phi(W)$ so that $\nu_V = \Phi\nu\Phi^{\mathsf{H}}$ and (5.20) means that, for all $\Psi \in \mathcal{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \nu_V)$ and $A \in \mathcal{A}$, $\int_A \Psi \, \mathrm{d}V = \int_A \Psi\Phi \, \mathrm{d}W$. Replacing Ψ by $\Psi \mathbb{1}_A$, it is sufficient to show this identity with $A = \Lambda$. Using that the integral with respect to a random g.o.s. measure is Gramian-isometric and Assertion (b) of Theorem 5.10, the mappings $\Psi \mapsto \int \Psi \, \mathrm{d}V$ and $\Psi \mapsto \int \Psi\Phi \, \mathrm{d}W$ are Gramian-isometric from $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu_W\Phi^{\mathsf{H}})$ to $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{I}_0, \mathbb{P})$. Hence by Theorem 5.4, they coincide on the whole space if they coincide on all $\Psi = \mathbb{1}_A \mathrm{Q}$ for $A \in \mathcal{A}$ and $\mathrm{Q} \in \mathcal{L}_b(\mathcal{G}_0, \mathcal{I}_0)$. To conclude the proof of Assertion (ii), it is thus enough to prove that, for all $A \in \mathcal{A}$ and $\mathrm{Q} \in \mathcal{L}_b(\mathcal{G}_0, \mathcal{I}_0)$, $\int_A \mathrm{Q} \, \mathrm{d}V = \int_A \mathrm{Q}\Phi \, \mathrm{d}W$. This identity follows from the definition of V and the fact that on both sides the operator Q can be moved in front of the integrals. This latter fact directly follows from the definition of the integral for the left-hand side and for the right-hand side when $\Phi = \mathbb{1}_B$ for some $B \in \mathcal{A}$, which extends to all Φ by observing that $\Phi \mapsto \int \mathrm{Q}\Phi \, \mathrm{d}W$ and $\Phi \mapsto \mathrm{Q} \int \Phi \, \mathrm{d}W$ are continuous on $\mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$.

Proof of Assertion (iii). Continuing with the setting of the proof of the previous point, we now suppose that Φ is injective $\|\nu_W\|_1$-a.e. Assertions (c) and (a) of Theorem 5.10 give that $\Phi^{-1} \in \mathscr{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{G}_0, \mathcal{H}_0), \nu_V)$ (*i.e.* $V \in \hat{\mathcal{S}}_{\Phi^{-1}}(\Omega, \mathcal{F}, \mathbb{P})$) and $\Phi^{-1}\nu_V \left(\Phi^{-1}\right)^{\mathsf{H}} = \nu_W$. Hence, writing Relation (5.20) with $\Psi = \Phi^{-1}$, we get $\hat{F}_{\Phi^{-1}}(V) = \hat{F}_{\Phi^{-1}\Phi}(W) = W$. Moreover, reversing the roles of W and V in assertion (i) gives the embedding $\mathcal{H}^{W, \mathcal{I}_0} \underset{\approx}{\subseteq} \mathcal{H}^{\hat{F}_\Phi(W), \mathcal{I}_0}$ which, with Assertion (i), allow us to conclude that $\mathcal{H}^{W, \mathcal{I}_0} \cong \mathcal{H}^{\hat{F}_\Phi(W), \mathcal{I}_0}$. □

We conclude this section with the proof of Proposition 4.8.

Proof of Proposition 4.8. Using the Gramian-unitary operator between the modular time domain and the modular spectral domain, this result is a direct application of Corollary 5.11 with $\Lambda = \hat{\mathrm{G}}$ and $\mathcal{A} = \mathcal{B}(\hat{\mathrm{G}})$ and $W = \hat{X}$. □

5.5 Proofs of Section 4.3

The goal of this section is to provide a proof of Lemma 4.9. Before that, let us recall essential facts about the diagonalization of compact positive operators. Let $\mathcal{H}_0$ be a separable Hilbert space of dimension $N \in \{1, \cdots, +\infty\}$, $(\Lambda, \mathcal{A})$ be a measurable space and $\Phi \in \mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}_0)$ such that for all $\lambda \in \Lambda$, $\Phi(\lambda) \in \mathcal{S}_1^+(\mathcal{H}_0)$. Then, in this case, for any $\lambda \in \Lambda$, $\Phi(\lambda)$ admits the eigendecomposition

$$\Phi(\lambda) = \sum_{0 \le n < N} \sigma_n(\lambda) \phi_n(\lambda) \otimes \phi_n(\lambda) \,, \tag{5.21}$$

where the series converges in operator norm and the family $(\phi_n(\lambda))_{0\leq n<N}$ is orthonormal. Moreover, we have

$$\mathrm{Tr}(\Phi(\lambda)) = \sum_{0\leq n<N} \sigma_n(\lambda) < +\infty \ .$$

The following theorem shows that such a decomposition can be constructed in a way which makes the eigenvalues and eigenvectors measurable as functions of λ. We recall that the weak topology on $\mathcal{H}_0$ is defined as the smallest topology which makes the functions $\left\{x^{\mathsf{H}} \,:\, x \in \mathcal{H}_0\right\}$ continuous.

Theorem 5.12. *Let $\mathcal{H}_0$ be a separable Hilbert space and $(\Lambda, \mathcal{A})$ be a measurable space. Let $\Phi \in \mathbb{F}_s(\Lambda, \mathcal{A}, \mathcal{H}_0)$ such that for all $\lambda \in \Lambda$, $\Phi(\lambda) \in \mathcal{S}_1^+(\mathcal{H}_0)$. Then the pairs $\{(\sigma_n, \phi_n) \,:\, 0 \leq n < N\}$ in* (5.21) *can be taken so that for all $0 \leq n < N$, σ_n is measurable from $(\Lambda, \mathcal{A})$ to $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ and ϕ_n is measurable from $(\Lambda, \mathcal{A})$ to $(\mathcal{H}_0, \mathcal{B}(\mathcal{H}_0))$.*

Proof. The construction of the eigenvalues and eigenvectors is done iteratively using the Measurable Maximum Theorem (see [1, Theorem 18.19]) on $\Lambda \times \bar{B}_{0,1}$, where $\bar{B}_{0,1}$ denotes the closed unit ball of $\mathcal{H}_0$, which is compact metrizable for the weak topology by the Banach-Alaoglu theorem and [1, Theorem 6.32]. As in [1, Definition 17.1], a *correspondence* φ from Λ to $\bar{B}_{0,1}$, denoted by $\varphi : \Lambda \twoheadrightarrow \bar{B}_{0,1}$, is a mapping which assigns each element of Λ to a subset of $\bar{B}_{0,1}$.

Construction of (σ_1, ϕ_1) : Define

$$f : \begin{array}{ccc} \Lambda \times \bar{B}_{0,1} & \to & \mathbb{R}_+ \\ (\lambda, x) & \mapsto & \langle \Phi(\lambda)x, x\rangle_{\mathcal{H}_0} \end{array} .$$

Then, for all $x \in \bar{B}_{0,1}$, the mapping $\lambda \mapsto f(\lambda, x)$ is measurable. Moreover, for all $\lambda \in \Lambda$, considering the eigendecomposition of $\Phi(\lambda)$, the mapping $x \mapsto f(\lambda, x)$ can be written as the uniform limit of continuous functions (for the weak topology on $\bar{B}_{0,1}$) hence it is continuous for the weak topology on $\bar{B}_{0,1}$. Finally, the correspondence

$$\varphi : \begin{array}{c} \Lambda \twoheadrightarrow \bar{B}_{0,1} \\ \lambda \mapsto \bar{B}_{0,1} \end{array}$$

is weakly measurable (in the sense of [1, Definition 18.1]) with nonempty compact values (for the weak topology). Therefore the Measurable Maximum Theorem [1, Theorem 18.19] gives that $m : \lambda \mapsto \max_{x\in\bar{B}_{0,1}} f(\lambda, x)$ is measurable and that there exists a function $g : \Lambda \to \bar{B}_{0,1}$ such that for all $\lambda \in \Lambda$, $g(\lambda) \in \mathrm{argmax}_{x\in\bar{B}_{0,1}} f(\lambda, x)$ and g is measurable from Λ to $\bar{B}_{0,1}$ endowed with the Borel σ-field $\mathcal{B}_w(\mathcal{H}_0)$ generated by the weak topology. This implies the usual measurability with respect to the σ-field $\mathcal{B}(\mathcal{H}_0)$ generated by the norm topology because, for all $x \in \mathcal{H}_0$, the mapping $y \mapsto \|x - y\|_{\mathcal{H}_0}$ is measurable from $(\mathcal{H}_0, \mathcal{B}_w(\mathcal{H}_0))$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. We set $\sigma_0 = m$ and $\phi_0 = g$. Then, from the definitions of f, m and g, that $\sigma_0(\lambda)$ is the largest eigenvalue of $\Phi(\lambda)$ and that $\phi_0(\lambda)$ is an eigenvector with eigenvalue $\sigma_0(\lambda)$.

Construction of (σ_n, ϕ_n) : Assume we have constructed n measurable functions $\sigma_0, \cdots, \sigma_{n-1}$ and $\phi_0, \cdots, \phi_{n-1}$ satisfying for all $\lambda \in \Lambda$, $\sigma_0(\lambda) \geq \cdots \geq \sigma_{n-1}(\lambda)$, and $(\phi_0(\lambda), \cdots, \phi_{n-1}(\lambda))$ is an orthonormal family where for all $0 \leq i \leq n-1$, $\phi_i(\lambda) \in \ker(\Phi(\lambda) - \sigma_i(\lambda)\mathrm{Id}_{\mathcal{H}_0})$. Then, as in the initialization step, the function

$$f : \begin{array}{ccc} \Lambda \times \bar{B}_{0,1} & \to & \mathbb{R}_+ \\ (\lambda, x) & \mapsto & \langle \Phi(\lambda)x, x\rangle_{\mathcal{H}_0} - \sum_{i=1}^{n-1} \sigma_i(\lambda) \left|\langle x, \phi_i(\lambda)\rangle_{\mathcal{H}_0}\right|^2 \end{array} .$$

is measurable in λ and continuous in x (for the weak topology) and the correspondence

$$\varphi : \begin{array}{ccc} \Lambda & \twoheadrightarrow & \bar{B}_{0,1} \\ \lambda & \mapsto & \bar{B}_{0,1} \cap \mathrm{Span}\left(\phi_0(\lambda), \cdots, \phi_{n-1}(\lambda)\right)^{\perp} \end{array}$$

is weakly measurable (in the sense of [1, Definition 18.1]) because of [1, Corollary 18.8 and Lemma 18.2]) and the fact that $\varphi(\lambda) = \left\{x \in \bar{B}_{0,1} \,:\, \sum_{i=0}^{n-1} \left|\langle x, \phi_i(\lambda)\rangle_{\mathcal{H}_0}\right|^2 = 0\right\}$ and has nonempty compact values (because $\varphi(\lambda)$ is a closed subset of $\bar{B}_{0,1}$ for the weak topology hence is compact for this topology). Hence, as previously, the Measurable Maximum Theorem gives that $m : \lambda \mapsto \max_{x\in\varphi(\lambda)} f(\lambda, x)$ is measurable and that there exists a measurable function $g : \Lambda \to \mathcal{H}_0$ such that for all $\lambda \in \Lambda$, $g(\lambda) \in \operatorname{argmax}_{x\in\varphi(\lambda)} f(\lambda, x)$. We set $\sigma_n = m$ and $\phi_n = g$. Then, from the definitions of f, m and g, we get that $\sigma_n(\lambda) \leq \sigma_{n-1}(\lambda)$ is the $(n+1)$-th largest eigenvalue of $\Phi(\lambda)$ (because it is the largest eigenvalue of $\Phi(\lambda) - \sum_{i=0}^{n-1} \sigma_i(\lambda)\phi_i(\lambda) \otimes \phi_i(\lambda)$) and that $\phi_n(\lambda)$ is an eigenvector with eigenvalue $\sigma_n(\lambda)$ and is orthogonal to $\phi_0, \cdots, \phi_{n-1}$. □

We can now prove Lemma 4.9.

Proof of Lemma 4.9. We provide a proof in the case where $N = \infty$ as the finite dimensional case is easier. Let $f \in L^1(\Lambda, \mathcal{A}, \mathcal{S}_1^+(\mathcal{H}_0), \mu)$ be the density of ν with respect to μ. We assume without loss of generality that $f(\lambda) \in \mathcal{S}_1(\mathcal{H}_0)^+$ for all $\lambda \in \hat{\mathrm{G}}$ (rather than for μ-almost every λ). Using Theorem 5.12 we can write

$$f(\lambda) = \sum_{n=0}^{+\infty} \sigma_n(\lambda)\phi_n(\lambda) \otimes \phi_n(\lambda) , \tag{5.22}$$

where $(\sigma_n(\lambda))_{n\in\mathbb{N}}$ is non-decreasing and converges to zero and $(\phi_n(\lambda))_{n\in\mathbb{N}}$ satisfies (ii). Moreover, for all $\lambda \in \Lambda$, $\sum_n \sigma_n(\lambda) = \|f(\lambda)\|_1 < \infty$, and we get Assertions (i) and (iii).

It only remains to prove (iv)–(vi), which we now proceed to do. By (5.22) and the previously proved assertions, we get that for all $n \in \mathbb{N}$ and all $\lambda \in \Lambda$, $\left\|\phi_n^{\mathsf{H}} f^{1/2}(\lambda)\right\|_2^2 = \sigma_n(\lambda) \leq \|f(\lambda)\|_1$. Hence $\phi_n^{\mathsf{H}} f^{1/2} \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0, \mathbb{C}), \nu)$ and Proposition 5.3 gives that $\phi_n^{\mathsf{H}} \in \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0, \mathbb{C}), \nu)$ and for all $n, p \in \mathbb{N}$,

$$\left\langle \phi_n^{\mathsf{H}}, \phi_p^{\mathsf{H}} \right\rangle_\nu = \int \phi_n^{\mathsf{H}} f \phi_p \, \mathrm{d}\mu = \begin{cases} 0 & \text{if } n \neq p, \\ \int \sigma_n \, \mathrm{d}\mu & \text{otherwise.} \end{cases} ,$$

where the last equality comes from (5.22) and the previously proved assertions.

Similarly, for all $n \in \mathbb{N}$, $\phi_n \otimes \phi_n f^{1/2} \in L^2(\Lambda, \mathcal{A}, \mathcal{S}_2(\mathcal{H}_0), \nu)$, hence by Proposition 5.3, we have $\phi_n \otimes \phi_n \in \mathsf{L}^2(\Lambda, \mathcal{A}, \mathcal{O}(\mathcal{H}_0), \nu)$ and for all $n, p \in \mathbb{N}$,

$$[\phi_n \otimes \phi_n, \phi_p \otimes \phi_p]_\nu = \int (\phi_n \otimes \phi_n) f (\phi_p \otimes \phi_p) \, \mathrm{d}\mu = \begin{cases} 0 & \text{if } n \neq p, \\ \int \sigma_n \, (\phi_n \otimes \phi_n) \, \mathrm{d}\mu & \text{otherwise,} \end{cases}$$

which proves Assertion (v). Now observe that, for all $\lambda \in \Lambda$, $\left(\sum_{n=0}^{\infty} \phi_n(\lambda) \otimes \phi_n(\lambda)\right) f(\lambda) = f(\lambda) \left(\sum_{n=0}^{\infty} \phi_n(\lambda) \otimes \phi_n(\lambda)\right) = f(\lambda)$. This yields $\left\|\sum_{n=0}^{\infty} \phi_n \otimes \phi_n - \mathrm{Id}_{\mathcal{H}_0}\right\|_\nu = 0$, and thus Assertion (vi) holds, which concludes the proof. □

A first consequence of Lemma 4.9 is the following.

Remark 5.1. *Applying Lemma 4.9 to the trace-class p.o.v.m. ν_X, we deduce that*

$$\hat{X} = \hat{F}_{\left(\sum_{0\leq n<N} \phi_n \otimes \phi_n\right)}(\hat{X}) = \sum_{0\leq n<N} \hat{F}_{\phi_n \otimes \phi_n}(\hat{X}) , \tag{5.23}$$

where $(\hat{F}_{\phi_n\otimes\phi_n}(\hat{X}))_{0\leq n<N}$ are uncorrelated random g.o.s.'s on $(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{H}_0)$. In other words, (4.10) holds both with $\hat{X}$ in the sum sign or out of it in the right-hand side. Moreover, for all $n \in \mathbb{N}$, $\hat{F}_{\phi_n\otimes\phi_n}(\hat{X}) = \hat{F}_{\phi_n} \circ \hat{F}_{\phi_n^{\mathsf{H}}}(\hat{X})$ and, by (iv) of Lemma 4.9, $(\hat{F}_{\phi_n^{\mathsf{H}}}(\hat{X}))_{0\leq n<N}$ is a sequence of uncorrelated $\mathbb{C}$-valued o.s. measures. Hence, interpreting (5.23) in the time domain, we get a decomposition of the process $X = (X_t)_{t\in\mathrm{G}}$ based on a collection of the uncorrelated univariate processes $(F_{\phi_n^{\mathsf{H}}}(X))_{0\leq n<N}$.

To conclude, we prove Proposition 4.10.

Proof of Proposition 4.10. Let $f_X(\chi) = \sum_{0\leq n<N} \sigma_n(\chi) \, \phi_n(\chi) \otimes \phi_n(\chi)$ denote the density of ν_X with respect to μ as given by Lemma 4.9. We have, for all $t \in \mathrm{G}$ and $\Theta \in \mathsf{L}^2(\hat{\mathrm{G}}, \mathcal{B}(\hat{\mathrm{G}}), \mathcal{O}(\mathcal{H}_0), \nu_X)$, $[F_\Theta(X)]_t = \int \chi(t) \, \Theta(\chi) \, \hat{X}(\mathrm{d}\chi)$, and thus by isometric isomorphism between the spectral domain and the time domain,

$$\mathbb{E}\left[\|X_t - [F_\Theta(X)]_t\|^2\right] = \int \left\|(\mathrm{Id}_{\mathcal{H}_0} - \Theta(\chi)) f_X^{1/2}(\chi)\right\|_2^2 \mu(\mathrm{d}\chi) .$$

The result is then obtained by observing that, for each $\chi \in \hat{\mathrm{G}}$, the norm in the integral is minimal under the constraint $\operatorname{rank}(\Theta(\chi)) \leq q(\chi)$ for $\Theta(\chi) = \sum_{0\leq n<q(\chi)\wedge N} \phi_n(\chi) \otimes \phi_n(\chi)$. □

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